Secondary Mathematics 2 Table of Contents

Unit 1

Extending the Number System

Unit 1 Cluster 1 (N.RN.1 & N.RN.2):
Extending Properties of Exponents

Cluster 1: Extending properties of exponents

1.1.1 Define rational exponents and extend the properties of integer exponents to rational exponents

1.1.2 Rewrite expressions, going between rational and radical form

VOCABULARY

If the exponent on a term is of the form $\frac{m}{n}$, where $n \neq 0$, then the number is said to have a **rational exponent.** $8^{\frac{1}{3}}$ is an example of a constant with a rational exponent.

Properties of Exponents (All bases are non-zero)	**Properties of Rational Exponents** (All bases are non-zero)	**Examples**
$x^a \cdot x^b = x^{a+b}$	$x^{\frac{p}{q}} \cdot x^{\frac{r}{s}} = x^{\frac{p}{q}+\frac{r}{s}}$	$x^{\frac{3}{4}} \cdot x^{\frac{1}{5}} = x^{\frac{3}{4}+\frac{1}{5}} = x^{\frac{19}{20}}$
$\frac{x^a}{x^b} = x^{a-b}$	$\frac{x^{\frac{p}{q}}}{x^{\frac{r}{s}}} = x^{\frac{p}{q}-\frac{r}{s}}$	$\frac{x^{\frac{2}{3}}}{x^{\frac{3}{5}}} = x^{\frac{2}{3}-\frac{3}{5}} = x^{\frac{1}{15}}$
$x^0 = 1$	Students should have this property memorized.	$\frac{x^a}{x^a} = x^{a-a} = x^0$ However: $\frac{x^a}{x^a} = 1$ Therefore: $x^0 = 1$
$x^{-a} = \frac{1}{x^a}$	$x^{-\frac{p}{q}} = \frac{1}{x^{\frac{p}{q}}}$	$x^{-\frac{2}{5}} = \frac{1}{x^{\frac{2}{5}}}$
$(xy)^n = x^n y^n$	$(xy)^{\frac{p}{q}} = x^{\frac{p}{q}} y^{\frac{p}{q}}$	$(xy)^{\frac{3}{4}} = x^{\frac{3}{4}} y^{\frac{3}{4}}$
$(x^m)^n = x^{m \cdot n}$	$\left(x^{\frac{p}{q}}\right)^{\frac{r}{s}} = x^{\frac{p}{q}\cdot\frac{r}{s}}$	$\left(x^{\frac{3}{4}}\right)^{\frac{2}{3}} = x^{\frac{3}{4}\cdot\frac{2}{3}} = x^{\frac{2}{4}} = x^{\frac{1}{2}}$

$\left(\frac{x}{y}\right)^m = \frac{x^m}{y^m}$	$\left(\frac{x}{y}\right)^{\frac{p}{q}} = \frac{x^{\frac{p}{q}}}{y^{\frac{p}{q}}}$	$\left(\frac{x}{y}\right)^{\frac{2}{3}} = \frac{x^{\frac{2}{3}}}{y^{\frac{2}{3}}}$
$\left(\frac{x^m}{y^n}\right)^{-1} = \frac{x^{-m}}{y^{-n}} = \frac{y^n}{x^m}$	$\left(\frac{x^{\frac{m}{n}}}{y^{\frac{a}{b}}}\right)^{-1} = \frac{x^{-\frac{m}{n}}}{y^{-\frac{a}{b}}} = \frac{y^{\frac{a}{b}}}{x^{\frac{m}{n}}}$	$\left(\frac{x^{\frac{1}{2}}}{y^{\frac{3}{4}}}\right)^{-1} = \frac{x^{-\frac{1}{2}}}{y^{-\frac{3}{4}}} = \frac{y^{\frac{3}{4}}}{x^{\frac{1}{2}}}$

Practice Exercises A

Simplify each expression using only positive exponents.

1. $5^{1/2} \cdot 5^{1/4}$
2. $\left(\frac{12^{1/3}}{4^{1/3}}\right)^2$
3. $\frac{1}{k^{-1/3}}$
4. $\left(8^{1/2} \cdot 5^{1/3}\right)^2$
5. $y^{-2/3}$
6. $z^{2/3} \cdot z^{1/2}$
7. $\left(2^4 \cdot 3^4\right)^{-1/4}$
8. $\left(4^{2/3}\right)^6$
9. z^0
10. $\frac{7}{7^{1/3}}$
11. $\frac{y^{2/3}}{y^{1/3}}$
12. $\frac{x^4}{x^{\frac{3}{7}}}$

Definition

A **radical** can also be written as a term with a rational exponent. For example, $x^{\frac{1}{n}} = \sqrt[n]{x}$ where n is an integer and $n \neq 0$. In general, $x^{\frac{m}{n}} = \sqrt[n]{x^m}$ where m and n are integers and $n \neq 0$.

$$x^{\frac{m}{n}} = \sqrt[n]{x^m}$$

The denominator of the rational exponent becomes the index of the radical.

$x^{\frac{m}{n}}$ can also be written as $\left(\sqrt[n]{x}\right)^m$

Rational Exponent Form	Radical Form
$x^{\frac{19}{20}}$	$\sqrt[20]{x^{19}}$
$x^{\frac{3}{2}}$	$\sqrt{x^3}$
$a^{\frac{7}{3}}$	$\sqrt[3]{a^7}$

Practice Exercises B

Rewrite each expression in radical form

1. $8^{4/3}$
2. $x^{3/4}$
3. $a^{5/9}$
4. $36^{3/2}$
5. $k^{3/2}$
6. $2x^{1/5}$
7. $(-64)^{2/3}$
8. $(-8)^{5/3}$
9. $(2x)^{1/5}$

Practice Exercises C

Rewrite each expression with rational exponents.

1. $\sqrt[3]{11}$
2. $\left(\sqrt[7]{42}\right)^2$
3. $\left(\sqrt[3]{-10}\right)^8$
4. $\left(\sqrt[4]{5}\right)^2$
5. $\sqrt[6]{x^7}$
6. $\sqrt[3]{r}$
7. $\sqrt[7]{w^5}$
8. $\sqrt[3]{k^2}$
9. $\left(\sqrt[5]{z}\right)^2$

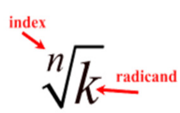
Vocabulary

For an integer n greater than 1, if $a^n = k$, then a is the ***n*th root of *k***.

A radical or the principal n^{th} root of *k*:

k, the radicand, is a real number.

n, the index, is a positive integer greater than one.



Properties of Radicals
$\sqrt[n]{a^n} = a$
$\sqrt[n]{ab} = \sqrt[n]{a} \cdot \sqrt[n]{b}$
$\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}$

Simplifying Radicals:

Radicals that are simplified have:

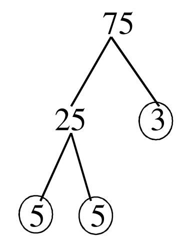
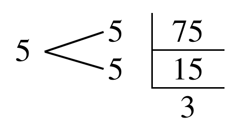
- no fractions left under the radical.
- no perfect power factors in the radicand, *k*.
- no exponents in the radicand, *k*, greater than the index, *n*.

Vocabulary	Example:
A **prime number** is a whole number greater than 1 that is only divisible by 1 and itself. In other words, a prime number has exactly two factors: 1 and itself.	$5 = 1 \cdot 5$ *prime* $30 = 2 \cdot 3 \cdot 5$ *not prime*

Division Rules For a Few Prime Numbers

A number is divisible by:	If:	Example:
2	The last digit is even (0, 2, 4, 6, 8)	256 is 255 is not
3	The sum of the digits is divisible by 3	381 (3+8+1=12 and 12÷3=4) Yes 383 (3+8+3=14 and $14 \div 3 = 4\frac{2}{3}$) No
5	The last digit is 0 or 5	175 is 809 is not
7	If you double the last digit and subtract it from the rest of the number and the answer is: • 0 • Divisible by 7	672 (Double 2 is 4, $67 - 4 = 63$ and $63 \div 7 = 9$) Yes 905 (Double 5 is 10, $90 - 10 = 80$ and $80 \div 7 = 11\frac{3}{7}$) No

Simplifying Radicals

Method 1: **Find Perfect Squares Under the Radical** 1. Rewrite the radicand as factors that are perfect squares and factors that are not perfect squares. 2. Rewrite the radical as two separate radicals. 3. Simplify the perfect square.	**Example:** $\sqrt{75}$ $\sqrt{25\cdot 3}$ $\sqrt{25}\cdot\sqrt{3}$ $5\sqrt{3}$
Method 2: **Use a Factor Tree** 1. Work with only the radicand. 2. Split the radicand into two factors. 3. Split those numbers into two factors until the number is a prime number. 4. Group the prime numbers into pairs. 5. List the number from each pair only once outside of the radicand. 6. Leave any unpaired numbers inside the radical. Note: If you have more than one pair, multiply the numbers outside of the radical as well as the numbers left inside.	**Example:** $\sqrt{75}$  $5\sqrt{3}$
Method 3: **Divide by Prime Numbers** 1. Work with only the radicand. 2. Using only prime numbers, divide each radicand until the bottom number is a prime number. 3. Group the prime numbers into pairs. 4. List the number from each pair only once outside of the radicand. 5. Leave any unpaired numbers inside the radical. Note: If you have more than one pair, multiply the numbers outside of the radical as well as the numbers left inside.	**Example:** $\sqrt{75}$  $5\sqrt{3}$

Method 4: **Use Exponent Rules**	**Example:** $\sqrt{75}$
1. Rewrite the exponent as a rational exponent.	$75^{1/2}$
2. Rewrite the radicand as factors that are perfect squares and factors that are not perfect squares.	$(25\cdot 3)^{1/2}$
3. Rewrite the perfect square factors with an exponent of 2.	$(5^2\cdot 3)^{1/2}$
4. Split up the factors, giving each the rational exponent.	$(5^2)^{1/2}\cdot(3)^{1/2}$
5. Simplify.	$5\cdot(3)^{1/2}$
6. Rewrite as a radical	$5\sqrt{3}$

Method 4 with Variables:	**Example:** $\sqrt[3]{x^7}$
1. Rewrite the exponent as a rational exponent.	$x^{7/3}$
2. Rewrite the radicand as two factors. One with the highest exponent that is divisible by the root and the other factor with an exponent of what is left over.	$(x^6\cdot x)^{1/3}$
3. Split up the factors, giving each the rational exponent.	$(x^6)^{1/3}\cdot x^{1/3}$
4. Rewrite the exponents using exponent rules.	$x^{6/3}\cdot x^{1/3}$
5. Simplify.	$x^2\cdot x^{1/3}$
6. Rewrite as a radical	$x^2\ \sqrt[3]{x}$

Practice D

Simplify each radical expression.

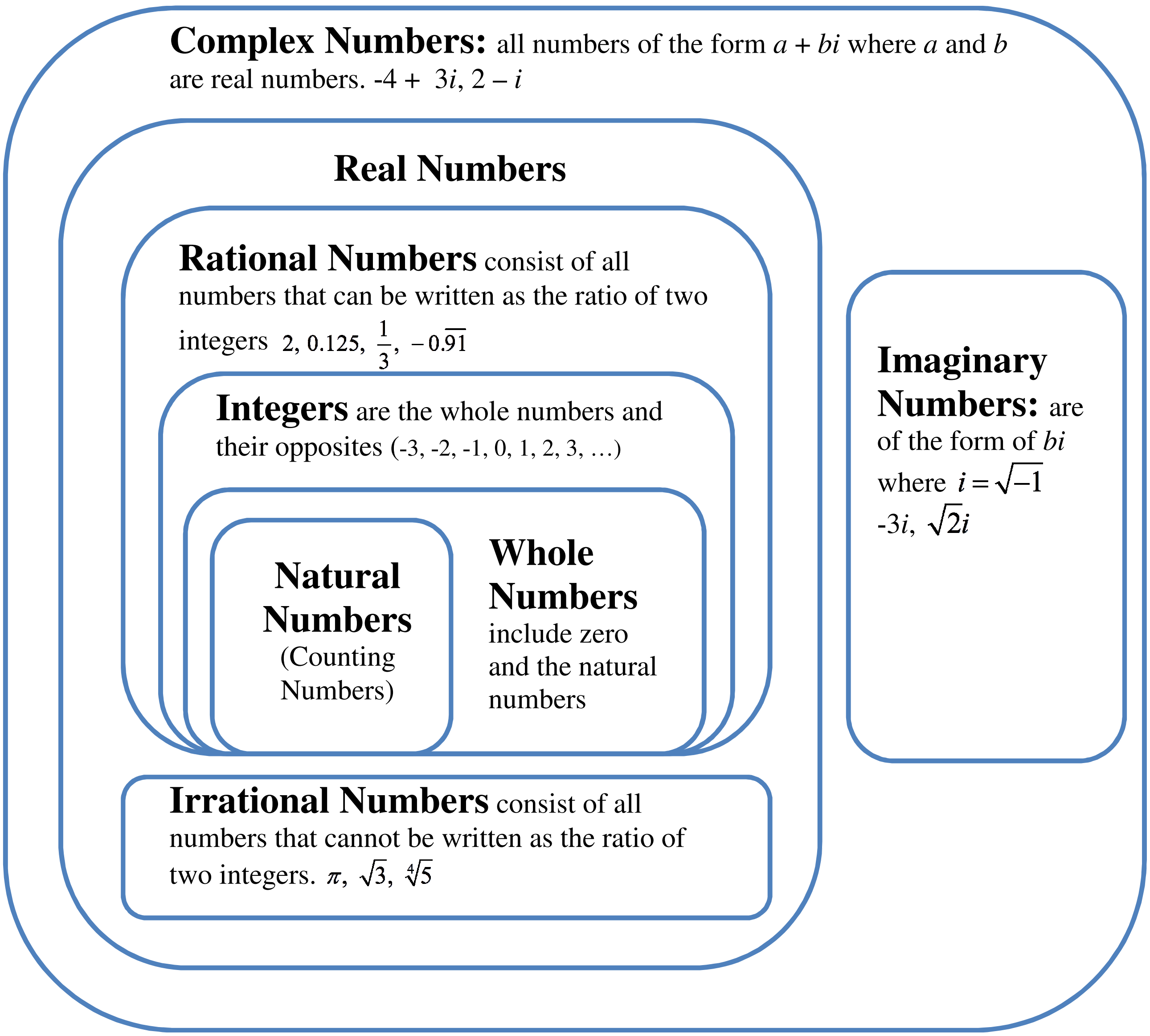
1. $\sqrt{45p^2}$
2. $\sqrt{80p^3}$
3. $\sqrt[3]{24x^3y^3}$
4. $\sqrt[3]{16u^4v^3}$
5. $\sqrt{75x^2y}$
6. $\sqrt[3]{64m^3n^3}$
7. $-4\sqrt{36y^3}$
8. $6\sqrt{150r}$
9. $7\sqrt[3]{96m^3}$

Unit 1 Cluster 2 (N.RN.3):
Using Properties of Rational and Irrational numbers

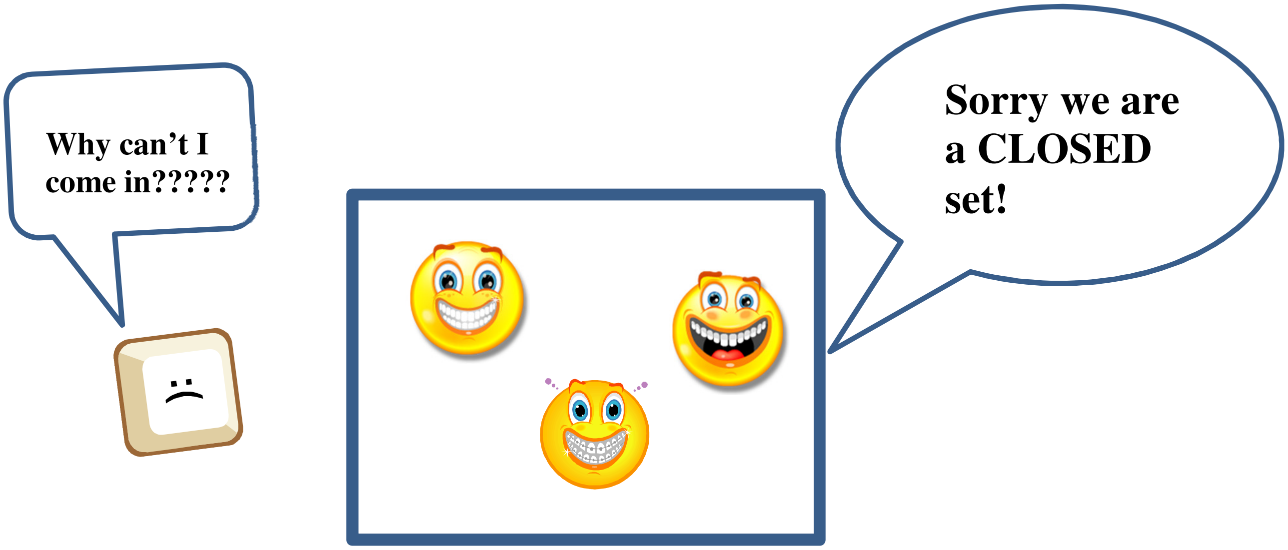
Cluster 1: Extending properties of exponents

1.2.1 Properties of rational and irrational numbers (i.e. sum of 2 rational numbers is rational, sum of a rational and irrational number is irrational)

Number Systems



Properties of Real Numbers		
Description	**Numbers**	**Algebra**
Commutative Property You can add or multiply real numbers in any order without changing the result.	$7+11=11+7$ $7(11)=11(7)$	$a+b=b+a$ $ab=ba$
Associative Property The sum or product of three or more real numbers is the same regardless of the way the numbers are grouped.	$(5+3)+7=5+(3+7)$	$(ab)c=a(bc)$
Distributive Property When you multiply a sum by a number, the result is the same whether you add and then multiply or whether you multiply each term by the number and then add the products.	$5(2+8)=5(2)+5(8)$ $(2+8)5=2(5)+2(8)$	$a(b+c)=ab+bc$ $(b+c)a=ba+ca$
Additive Identity Property The sum of a number and 0, the additive identity, is the original number.	$3+0=3$	$n+0=0+n=n$
Multiplicative Identity Property The product of a number and 1, the multiplicative identity, is the original number.	$\frac{2}{3}\cdot 1=\frac{2}{3}$	$n\cdot 1=1\cdot n=n$
Additive Inverse Property The sum of a number and its opposite, or additive inverse, is 0.	$5+(-5)=0$	$n+(-n)=0$
Multiplicative Inverse Property The product of a non-zero number and its reciprocal, or multiplicative inverse	$8\cdot\frac{1}{8}=1$	$n\cdot\frac{1}{n}=1,\ n\neq 0$
Closure Property The sum or product of any two real numbers is a real number.	$2+3=5$ $2(6)=12$	$a+b\in\mathbb{R}$ $ab\in\mathbb{R}$



Closure When an operation is executed on the members of a set, the result is guaranteed to be in the set.	
Addition: If two integers are added together, the sum is an integer. Therefore, integers are closed under addition.	**Example:** $-2+5=3$
Multiplication: If two integers are multiplied together, the product is an integer. Therefore, integers are closed under multiplication.	**Example:** $(-6)(7)=-42$
Subtraction: If one integer is subtracted from another, the difference is an integer. Therefore, integers are closed under subtraction.	**Example:** $-2-(-6)=4$
Division: If one integer is divided by another integer, the quotient may or may not be an integer. Therefore, integers are not closed under division.	**Example:** $10\div(-2)=-5$ *closed* $(-2)\div 10=-\frac{1}{5}$ *not closed*

You Decide

1. What number systems are closed under addition? Justify your conclusions using the method of your choice.

2. What number systems are closed under multiplication? Justify your conclusions using the method of your choice.

3. What number systems are closed under subtraction? Justify your conclusions using the method of your choice.

4. What number systems are closed under division? Justify your conclusions using the method of your choice.

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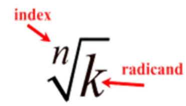

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Simplifying Radicals

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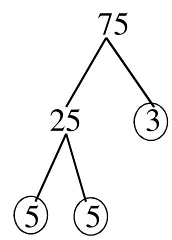
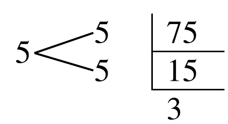
Method 2: **Use a Factor Tree** 7. Work with only the radicand. 8. Split the radicand into two factors. 9. Split those numbers into two factors until the number is a prime number. 10. Group the prime numbers into pairs. 11. List the number from each pair only once outside of the radicand. 12. Leave any unpaired numbers inside the radical. Note: If you have more than one pair, multiply the numbers outside of the radical as well as the numbers left inside.	**Example:** $\sqrt{75}$  $5\sqrt{3}$
Method 3: **Divide by Prime Numbers** 6. Work with only the radicand. 7. Using only prime numbers, divide each radicand until the bottom number is a prime number. 8. Group the prime numbers into pairs. 9. List the number from each pair only once outside of the radicand. 10. Leave any unpaired numbers inside the radical. Note: If you have more than one pair, multiply the numbers outside of the radical as well as the numbers left inside.	**Example:** $\sqrt{75}$  $5\sqrt{3}$

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9. Split up the factors, giving each the rational exponent.	$(x^6)^{1/3}\cdot x^{1/3}$
10. Rewrite the exponents using exponent rules.	$x^{6/3}\cdot x^{1/3}$
11. Simplify.	$x^2\cdot x^{1/3}$
12. Rewrite as a radical	$x^2\ \sqrt[3]{x}$

Adding and Subtracting Radicals *To add or subtract radicals, simplify first if possible, and then add or subtract "like" radicals.*	
1. They both have the same term under the radical so they are "like" terms. 2. Add the coefficients of the radicals.	**Example:** $2\sqrt{3}+5\sqrt{3}$ $(2+5)\sqrt{3}$ $7\sqrt{3}$
1. They both have the same term under the radical so they are "like" terms. 2. Subtract the coefficients of the radicals.	**Example:** $4\sqrt{3}-7\sqrt{3}$ $(4-7)\sqrt{3}$ $(-3)\sqrt{3}$

1. They are not "like" terms, but one of them can be simplified. 2. Rewrite the number under the radical. 3. Use the properties of radicals to write the factors as two radicals. 4. 25 is a perfect square and the square root of it is 5. 5. Multiply the coefficients of the second radical. 6. Now they are "like" terms, add the coefficients.	**Example:** $5\sqrt{3}+2\sqrt{75}$ $5\sqrt{3}+2\sqrt{25\cdot 3}$ $5\sqrt{3}+2\sqrt{25}\sqrt{3}$ $5\sqrt{3}+2\cdot 5\sqrt{3}$ $5\sqrt{3}+10\sqrt{3}$ $(5+10)\sqrt{3}$ $15\sqrt{3}$
1. None of them are "like" terms. Simplify if you can. 2. Factor each number inside the radical. 3. Use the properties of radicals to simplify. 4. 4 and 9 are perfect squares; their square roots are 2 and 3. 5. Multiply the numbers outside of the radical. 6. Only the terms with $\sqrt{2}$ are "like" terms. 7. Simplify.	**Example:** $5\sqrt{8}-3\sqrt{18}+\sqrt{3}$ $5\sqrt{4\cdot 2}-3\sqrt{9\cdot 2}+\sqrt{3}$ $5\sqrt{4}\sqrt{2}-3\sqrt{9}\sqrt{2}+\sqrt{3}$ $5\cdot 2\sqrt{2}-3\cdot 3\sqrt{2}+\sqrt{3}$ $10\sqrt{2}-9\sqrt{2}+\sqrt{3}$ $(10-9)\sqrt{2}+\sqrt{3}$ $\sqrt{2}+\sqrt{3}$
1. They are not like terms, but they can be simplified. 2. Rewrite the expressions under the radical. 3. Use properties of radicals to rewrite the expressions. 4. The cube root of 8 is 2 and the cube root of 27 is 3. 5. Multiply the coefficients of the last radical. 6. Add or subtract the coefficients of the like terms. 7. Simplify.	**Example:** $\sqrt[3]{40}-3\sqrt[3]{5}+2\sqrt[3]{135}$ $\sqrt[3]{5\cdot 8}-3\sqrt[3]{5}+2\sqrt[3]{5\cdot 27}$ $\sqrt[3]{5}\cdot\sqrt[3]{8}-3\sqrt[3]{5}+2\sqrt[3]{5}\cdot\sqrt[3]{27}$ $\sqrt[3]{5}\cdot 2-3\sqrt[3]{5}+2\cdot 3\sqrt[3]{5}$ $2\sqrt[3]{5}-3\sqrt[3]{5}+6\sqrt[3]{5}$ $(2-3+6)\sqrt[3]{5}$ $5\sqrt[3]{5}$

1. They are not like terms, but one of them can be simplified. 2. Rewrite the expression under the radical. 3. Use properties of radicals to rewrite the expression. 4. 8 and x^3 are perfect cubes. The cube root of 8 is 2 and x^3 is *x*. 5. Multiply the coefficients of the first radical. 6. Now they are like terms, add the coefficients of each. 7. Simplify.	**Example:** $2\sqrt[3]{24x^3} - x\sqrt[3]{3}$ $2\sqrt[3]{8\cdot 3\cdot x^3} - x\sqrt[3]{3}$ $2\cdot\sqrt[3]{8}\cdot\sqrt[3]{3}\cdot\sqrt[3]{x^3} - x\sqrt[3]{3}$ $2\cdot 2\cdot x\sqrt[3]{3} - x\sqrt[3]{3}$ $4x\sqrt[3]{3} - x\sqrt[3]{3}$ $(4x - x)\sqrt[3]{3}$ $3x\sqrt[3]{3}$

Multiplying Radicals **Multiplying radicals with the same index**	
1. Multiply the coefficients and multiply the numbers under the radicand. 2. If possible, simplify. This is already simplified.	**Example:** $2\sqrt{6}\cdot 3\sqrt{7}$ $(2\cdot 3)\sqrt{6\cdot 7}$ $6\sqrt{42}$
1. Use the distributive method to multiply. 2. Use properties of radicals to simplify. 3. Simplify any radicals. 4. Combined "like" terms if possible.	**Example:** $\left(\sqrt{3}+2\right)\left(\sqrt{6}-3\right)$ $\sqrt{3}\left(\sqrt{6}\right)+\sqrt{3}(-3)+2\sqrt{6}+2(-3)$ $\sqrt{18}-3\sqrt{3}+2\sqrt{6}-6$ $\sqrt{9\cdot 2}-3\sqrt{3}+2\sqrt{6}-6$ $3\sqrt{2}-3\sqrt{3}+2\sqrt{6}-6$
1. Use the distributive method to multiply. 2. Use properties of radicals to simplify. 3. Simplify and combine "like" terms. 4. The square root of 25 is 5. 5. Combine like terms.	**Example:** $\left(\sqrt{5}-1\right)\left(\sqrt{5}-4\right)$ $\sqrt{5}\cdot\sqrt{5}+\sqrt{5}(-4)+(-1)\sqrt{5}+(-1)(-4)$ $\sqrt{5\cdot 5}-4\sqrt{5}-\sqrt{5}+4$ $\sqrt{25}+(-4-1)\sqrt{5}+4$ $5+-5\sqrt{5}+4$ $9-5\sqrt{5}$

	Example:
	$\left(5\sqrt{2}+3\right)\left(5\sqrt{2}-3\right)$
1. Use the distributive method to multiply.	$5\sqrt{2}\cdot 5\sqrt{2}+5\sqrt{2}(-3)+\left(3\right)5\sqrt{2}+3(-3)$
2. Use properties of radicals to simplify.	$5\cdot 5\sqrt{2\cdot 2}-3\cdot 5\sqrt{2}+3\cdot 5\sqrt{2}-9$
3. Simplify and combine like terms.	$25\sqrt{4}+(-15+15)\sqrt{2}-9$
4. The square root of 4 is 2.	$25\cdot 2-9$
5. Simplify.	$50-9$
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Multiplying Radicals with Different Indices

Note: In order to multiply radicals with different indices the radicands must be the same.

	Example:
	$\sqrt{7}\cdot\sqrt[5]{7}$
1. Rewrite each radical using rational exponents.	$7^{\frac{1}{2}}\cdot 7^{\frac{1}{5}}$
2. Use properties of exponents to simplify.	$7^{\frac{1}{2}+\frac{1}{5}}$
3. Combine the fractions by finding a common denominator.	$7^{\frac{7}{10}}$

	Example:
	$\sqrt[3]{x^2}\cdot\sqrt[4]{x}$
1. Rewrite each radical using rational exponents.	$x^{\frac{2}{3}}\cdot x^{\frac{1}{4}}$
2. Use properties of exponents to simplify.	$x^{\frac{2}{3}+\frac{1}{4}}$
3. Combine the fractions by finding a common denominator.	$x^{\frac{11}{12}}$

	Example:
	$\sqrt{\sqrt[3]{x}}$
1. Rewrite the inner radical using rational exponents.	$\sqrt{x^{\frac{1}{3}}}$
2. Rewrite the outer radical using rational exponents.	$\left(x^{\frac{1}{3}}\right)^{\frac{1}{2}}$
3. Use properties of exponents to simplify.	$x^{\frac{1}{3}\cdot\frac{1}{2}}$
4. Simplify by multiplying fractions.	$x^{\frac{1}{6}}$

Practice Exercises A

Add or Subtract

1. $-10\sqrt{7}+12\sqrt{7}$
2. $2\sqrt[3]{3}-2\sqrt[3]{24}$
3. $-\sqrt{12}+3\sqrt{3}$
4. $3\sqrt[3]{2}-\sqrt[3]{54}$
5. $3\sqrt[3]{16}+3\sqrt[3]{2}$
6. $-3\sqrt{20}-\sqrt{5}$
7. $\sqrt[3]{40}-2\sqrt{6}-3\sqrt[3]{5}$
8. $-3\sqrt{18}+3\sqrt{8}-\sqrt{24}$
9. $2\sqrt{18}-2\sqrt{12}+2\sqrt{18}$

Practice Exercises B

Multiply and simplify the result.

1. $-4\sqrt{28x}\cdot\sqrt{7x^3}$
2. $3\sqrt[3]{12}\cdot\sqrt[3]{6}$
3. $\sqrt[3]{20x^2}\cdot 4\sqrt[3]{20x}$
4. $\sqrt{6}\left(\sqrt{3}+\sqrt{12}\right)$
5. $\sqrt[3]{4}\left(\sqrt[3]{2}-\sqrt[3]{5}\right)$
6. $\sqrt{3}\left(-5\sqrt{10}+\sqrt{6}\right)$
7. $\left(\sqrt{5}-\sqrt{3}\right)\left(\sqrt{5}+\sqrt{3}\right)$
8. $\left(-2\sqrt{3}+2\right)\left(\sqrt{3}-5\right)$
9. $\left(5-4\sqrt{5}\right)\left(-2+\sqrt{5}\right)$
10. $\sqrt[3]{6}\cdot\sqrt[5]{6}$
11. $\sqrt[4]{2y}\cdot\sqrt[7]{2y}$
12. $\sqrt[3]{\sqrt[8]{z}}$

You Decide:

1. Add: $2+\frac{4}{5}$. Can you write the result as the ratio of two numbers? (Use your graphing calculator to change the sum from a decimal to a fraction by pushing the math button and select FRAC)

2. Add: $\frac{1}{2}+\frac{2}{3}$. Can you write the result as the ratio of two numbers?

3. Add: $2+1.5$. Can you write the result as the ratio of two numbers?

4. Add: $1.75+1.35$. Can you write the result as the ratio of two numbers?

5. Add: $2+\sqrt{3}$. Can you write the result as the ratio of two numbers?

6. Add: $\frac{5}{3}+\pi$. Can you write the result as the ratio of two numbers?

7. Write a rule based on your observations with adding rational and irrational numbers.

Unit 1 Cluster 4 (A.APR.1): Polynomials

Cluster 4: Perform arithmetic operations on polynomials

1.4.1 Polynomials are closed under addition, subtraction, and multiplication

1.4.1 Add, subtract, and multiply polynomials (**NO DIVISION**)

VOCABULARY

A term that does not have a variable is called a **constant.** For example the number 5 is a **constant** because it does not have a variable attached and will always have the value of 5.

A constant or a variable or a product of a constant and a variable is called a **term**. For example 2, x, or $-3x^2$ are all terms.

Terms with the same variable to the same power are **like terms**. $2x^2$ and $-7x^2$ are like terms.

An expression formed by adding a finite number of unlike terms is called a **polynomial**. The variables can only be raised to positive integer exponents. $4x^3-6x^2+1$ is a polynomial, while $x^{\frac{3}{2}}-2x^{-1}+5$ is not a polynomial. **NOTE:** There are no square roots of variables, no fractional powers, and no variables in the denominator of any fractions.

A polynomial with only one term is called a **monomial** $\left(6x^4\right)$. A polynomial with two terms is called a **binomial** ($2x+1$). A polynomial with three terms is called a **trinomial** $\left(5x^2-x+3\right)$**.**

Polynomials are in **standard (general) form** when written with exponents in descending order and the constant term last. For example $2x^4-5x^3+7x^2-x+3$ is in standard form.

The exponent of a term gives you the **degree** of the term. The term $-3x^2$ has degree two. For a polynomial, the value of the *largest exponent* is the **degree** of the whole polynomial. The polynomial $2x^4-5x^3+7x^2-x+3$ has degree 4.

The number part of a term is called the **coefficient** when the term contains a variable and a number. $6x$ has a coefficient of 6 and $-x^2$ has a coefficient of -1.

The **leading coefficient** is the coefficient of the first term when the polynomial is written in standard form. 2 is the leading coefficient of $2x^4-5x^3+7x^2-x+3$.

Degree n

General Polynomial: $f(x)=a_nx^n+a_{n-1}x^{n-1}+\cdots+a_2x^2+a_1x+a_0$

Leading Coefficient a_n

Leading Term

Constant

CLASSIFICATIONS OF POLYNOMIALS

Name	Form	Degree	Example
Zero	$f(x)=0$	None	$f(x)=0$
Constant	$f(x)=a, a\neq 0$	0	$f(x)=5$
Linear	$f(x)=ax+b$	1	$f(x)=-2x+1$
Quadratic	$f(x)=ax^2+bx+c$	2	$f(x)=3x^2+\frac{1}{2}x+\frac{7}{9}$
Cubic	$f(x)=ax^3+bx^2+cx+d$	3	$f(x)=x^3-3x^2$

Practice Exercises A:

Determine which of the following are polynomial functions. If the function is a polynomial, state the degree and leading coefficient. If it is not, explain why.

1. $f(x)=3x^{-5}+17$
2. $f(x)=-9+2x$
3. $f(x)=2x^5-\frac{1}{2}x+9$
4. $f(x)=13$
5. $f(x)=\sqrt[3]{27x^3+8x^6}$
6. $f(x)=4x-5x^2$

Operations of Polynomials

Addition/Subtraction: Combine like terms.

Example 1:

Horizontal Method	Vertical Method
$\left(2x^3-3x^2+4x-1\right)+\left(x^3+2x^2-5x+3\right)$ $=\left(2x^3+x^3\right)+\left(-3x^2+2x^2\right)+\left(4x-5x\right)+\left(-1+3\right)$ $=3x^3-x^2-x+2$	$\begin{array}{r} 2x^3-3x^2+4x-1 \\ +\quad x^3+2x^2-5x+3 \\ \hline 3x^3-\ x^2-\ x+2 \end{array}$

Example 2:

Horizontal Method	Vertical Method
$\left(4x^2+3x-4\right)-\left(2x^3+x^2-x+2\right)$ $=4x^2+3x-4-2x^3-x^2+x-2$ $=-2x^3+3x^2+4x-6$	$\begin{array}{r} 4x^2+3x-4 \\ -\left(2x^3+x^2-\ x+2\right) \\ \hline -2x^3+3x^2+4x-6 \end{array}$

Multiplication: Multiply by a monomial

Example 3:

$$3x\left(2x^2+6x-5\right)$$
$$=3x\left(2x^2\right)+3x\left(6x\right)+3x\left(-5\right)$$
$$=6x^3+18x^2-15x$$

Example 4:

$$-5x^2\left(3x^3-2x^2+6x-8\right)$$
$$=-15x^5+10x^4-30x^3+40x^2$$

Multiplication: Multiply two binomials $(5x-7)(2x+9)$

Distributive (FOIL) Method	Box Method	Vertical Method
$(5x-7)(2x+9)$ $=5x(2x+9)-7(2x+9)$ $=10x^2+45x-14x-63$ **combine like terms* $=10x^2+31x-63$	$\begin{array}{c\|c\|c} & 5x & -7 \\ \hline 2x & 10x^2 & -14x \\ \hline 9 & 45x & -63 \end{array}$ **combine terms on the diagonals of the unshaded boxes(top right to lower left)* $=10x^2+31x-63$	$\begin{array}{rr} & 5x-7 \\ \times & 2x+9 \\ \hline & 45x-63 \\ 10x^2-14x & \\ \hline 10x^2+31x-63 & \end{array}$

Multiplication: Multiply a binomial and a trinomial $(2x+3)\left(6x^2-7x-5\right)$

Distributive Method	Box Method	Vertical Method
$(2x+3)\left(6x^2-7x-5\right)$ $=2x\left(6x^2-7x-5\right)+3\left(6x^2-7x-5\right)$ $=\left(12x^3-14x^2-10x\right)+\left(18x^2-21x-15\right)$ $=12x^3-14x^2-10x+18x^2-21x-15$ **combine like terms* $=12x^3+4x^2-31x-15$	$\begin{array}{c\|c\|c\|c} & 6x^2 & -7x & -5 \\ \hline 2x & 12x^3 & -14x^2 & -10x \\ \hline 3 & 18x^2 & -21x & -15 \end{array}$ **combine terms on the diagonals of the unshaded boxes(top right to lower left)* $=12x^3+4x^2-31x-15$	$\begin{array}{rrrr} & 6x^2 & -7x & -5 \\ \times & & 2x & +3 \\ \hline & 18x^2 & -21x & -15 \\ 12x^3 & -14x^2 & -10x & \\ \hline 12x^3 & +4x^2 & -31x & -15 \end{array}$

Practice Exercises B:

Perform the required operations. Write your answers in standard form and determine if the result is a polynomial.

1. $\left(x^2-3x+7\right)+\left(3x^2+5x-3\right)$
2. $\left(-3x^2-5\right)-\left(x^2+7x+12\right)$
3. $\left(4x^3-x^2+3x\right)-\left(x^3+12x-3\right)$
4. $-\left(y^2+2y-3\right)+\left(5y^2+3y+4\right)$
5. $2x\left(x^2-x+3\right)$
6. $y^2\left(2y^2+3y-4\right)$
7. $-3u\left(4u-1\right)$
8. $\left(2-x-3x^2\right)\left(5x\right)$
9. $\left(x+7\right)\left(x-3\right)$
10. $\left(3x-5\right)\left(x+2\right)$
11. $\left(2x+3\right)\left(4x+1\right)$
12. $\left(3x-y\right)\left(3x+y\right)$
13. $\left(2x+7\right)^2$
14. $\left(3-5x\right)^2$
15. $\left(5x^3-1\right)^2$
16. $\left(2x^3-3y\right)\left(2x^3+3y\right)$
17. $\left(x^2-2x+3\right)\left(x+4\right)$
18. $\left(x^2+3x-2\right)\left(x-3\right)$
19. $\left(x^2+x-3\right)\left(x^2+x+1\right)$
20. $\left(2x^2-3x+1\right)+\left(x^2+x+1\right)$

YOU DECIDE

Are polynomials closed under addition, subtraction, multiplication? Justify your conclusion using the method of your choice.

Unit 2

Quadratic Functions and Modeling

Unit 2 Cluster 1 (F.1F.4, F.1F.5, F.1F.6)
Unit 2 Cluster 2 (F.1F.7, F.1F.9)
Interpret functions that arise in applications in terms of a context
Analyzing functions using different representations

Cluster 1:

2.1.1 Interpret key features; intercepts, intervals where increasing and decreasing, intervals where positive and negative, relative maximums and minimums, symmetry, end behavior, domain and range, periodicity

2.1.2 Relate the domain of a function to its graph or a context

2.1.3 Average rate of change over an interval: calculate, interpret, and estimate from a graph.

Cluster 2:

2.2.1 Graph functions from equations by hand and with technology showing key features (square roots, cube roots, piecewise-defined functions including step and functions, and absolute value).

2.2.1 Graph linear and quadratic functions and show intercepts, maxima, and minima

2.2.3 Compare properties (key features) of functions each represented differently (table, graph, equation or description)

VOCABULARY

The **domain** is the set of all first coordinates when given a table or a set of ordered pairs. It is the set of all x-coordinates of the points on the graph and is the set of all numbers for which an equation is defined. The domain it is written from the least value to the greatest value.

The **range** is the set of all second coordinates when given a table or a set of ordered pairs. It is the set of all y-coordinates of the points on the graph. When modeling real world situations, the **range** is the set of all numbers that make sense in the problem. The range is written from the least value to the greatest value.

Example:

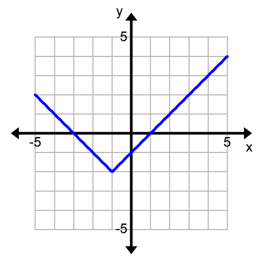
Find the domain and range of $f\left(x\right)=2\sqrt{x+2}-3$.

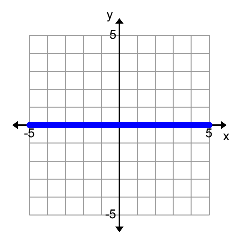
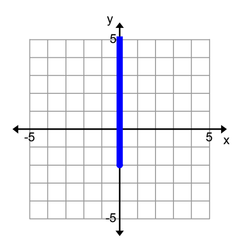
Domain	
1. Find any values for which the function is undefined.	The square root function has real number solutions if the expression under the radicand is positive or zero. This means that $x+2\geq 0$ therefore $x\geq -2$.
2. Write the domain in interval notation.	The domain is $[-2,\infty)$.

Range	
1. Find all values for which the output exists.	The square root function uses the principal square root which is a positive number or zero ($y \geq 0$). However, the function has been shifted down three units so the range is also shifted down three units $y \geq -3$.
2. Write the range in interval notation.	The range is $[-3,\infty)$

Example:

Find the domain and range of the function graphed to the right.



Domain	
1. List all the *x*-values of the function graphed.	If you were to flatten the function against the *x*-axis you would see something like this:  The function is defined for all the *x*-values along the *x*-axis.
2. Write the domain in interval notation.	The domain is $\left(-\infty,\infty\right)$.
Range	
1. List all the *y*-values of the function graphed.	If you were to flatten the function against the y-axis you would see something like this:  The function is defined for all the *y*-values greater than or equal to -2.
2. Write the range in interval notation.	The range is $[-2,\infty)$.

Example:

The path of a ball thrown straight up can be modeled by the equation $h(t) = -16t^2 + 20t + 4$ where t is the time in seconds that the ball is in the air and h is the height of the ball. What is the real world domain and range for the situation?

Domain 1. Find all the values that would make sense for the situation.	The domain represents the amount of time that the ball is in the air. At $t = 0$ the ball is thrown and enters the air shortly afterwards so the domain must be greater than zero. The ball will hit the ground at 1.425 seconds. Once it is on the ground it is no longer in the air so the domain must be less than 1.425 seconds. The ball is in the air for $0 < t < 1.425$ seconds.
2. Write the domain in interval notation.	The domain is $(0, 1.425)$.
Range 1. Find all the values that would make sense for the situation.	The ball will not go lower than the ground so the height must be greater than zero. The ball will go no higher than its maximum height so the height must be less than or equal to 10.25 feet. The range will be $0 < h \leq 10.25$.
2. Write the range in interval notation.	The range is $(0, 10.25]$.

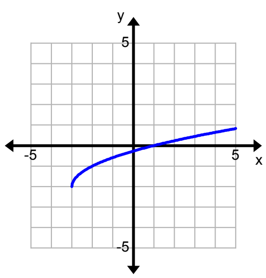
Practice Exercises A:

Find the domain and range.

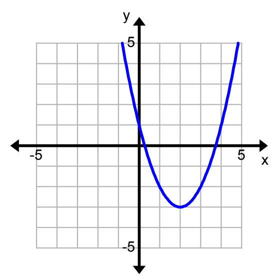
1. $f(x) = 3x + 2$

2. $f(x) = \sqrt[3]{x-1}$

3.



4.



5. Your cell phone plan charges a flat fee of \$10 for up to1000 texts and \$0.10 per text over 1000.

6. The parking lot for a movie theater in the city has no charge for the first hour, but charges \$1.50 for each additional hour or part of an hour with a maximum charge of \$7.50 for the night.

VOCABULARY

The ***x*-intercept** is where a graph crosses or touches the x-axis. It is the ordered pair $(a,0)$. Where a is a real number.

The ***y* -intercept** is where a graph crosses or touches the y-axis. It is the ordered pair $(0,b)$. Where b is a real number.

A **relative maximum** occurs when the y-value is greater than all of the y-values near it. A function may have more than one relative maximum value. A **relative minimum** occurs when the y-value is less than all of the y-values near it. A function may have more than one relative minimum value.

Example:

Find the intercepts of the function $f(x)=2x-1$.

***x*-intercept**	
1. Substitute y in for $f(x)$.	$y=2x-1$
2. Substitute 0 in for y.	$0=2x-1$
3. Solve for x.	$1=2x$
	$\frac{1}{2}=x$
4. Write the intercept as an ordered pair.	$\left(\frac{1}{2},0\right)$
***y*-intercept**	
1. Substitute 0 in for x.	$y=2(0)-1$
2. Solve for y.	$y=0-1$
	$y=-1$
3. Write the intercept as an ordered pair.	$(0,-1)$

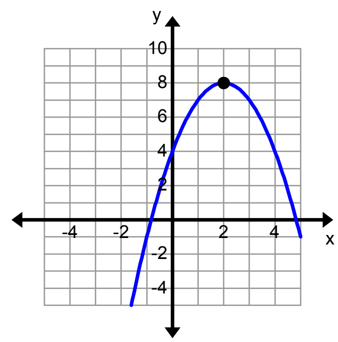
Example:

Find the intercepts of the function $f(x) = 3x^2 - 5x - 2$.

***x*-intercept** 1. Use your graphing calculator to graph the function. 2. Use the Calculate Menu (2^{nd}, Trace, Zero) to find the *x*-intercepts. (Zero is another name for the *x*-intercept)	 The *x*-intercepts are $\left(-\frac{1}{3}, 0\right)$ and $(2,0)$.
***y*-intercept** 1. To find the y-intercept, replace each *x* with 0. 2. Solve the equation for *y*. 3. Write the intercept as an ordered pair.	$y = 3(0)^2 - 5(0) - 2$ $y = 0 - 0 - 2$ $y = -2$ $(0,2)$

Example:

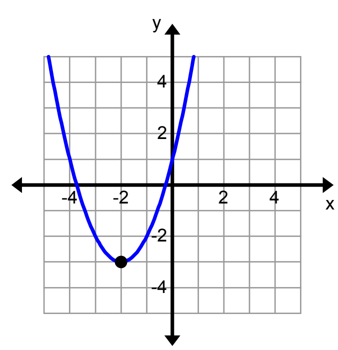
Find the maximum of $f(x) = -x^2 + 4x + 4$.



To find the maximum use your graphing calculator to graph the function. Then use the Calculate Menu (2^{nd}, Trace, Maximum). Enter a number that is to the left of the maximum, for example 0, then push enter. Then enter a number that is to the right of the maximum, for example 4, then push enter. You can guess the value of the maximum or just push enter again and the maximum will be calculated. The maximum is (2, 8).

Example:

Find the minimum of $f(x) = (x+2)^2 - 3$.



To find the maximum use your graphing calculator to graph the function. Then use the Calculate Menu (2^{nd}, Trace, Minimum). Enter a number that is to the left of the minimum, for example -3, then push enter. Then enter a number that is to the right of the minimum, for example -1, then push enter. You can guess the value of the minimum or just push enter again and the minimum will be calculated. The minimum is (-2, -3).

Practice Exercises B

Find the x and y-intercepts for each function.

1. $2x+5y=10$
2. $f(x)=-4x+7$
3. $f(x)=x^2-x-30$
4. $f(x)=\frac{1}{3}x-4$
5. $f(x)=x^2+3x-18$
6. $f(x)=-2x^2-3x+4$

Find the relative maximums or minimums of each function.

7. $f(x)=-2(x-1)^2+3$
8. $f(x)=3(x-2)^2+7$
9. $f(x)=-4x^2-16x-18$
10. $f(x)=(x-5)^2-4$
11. $f(x)=3x^2-18x+23$
12. $f(x)=x^2-8x+14$

VOCABULARY

An **interval** is a set of numbers between two x-values. An **open interval** is a set of numbers between two x-values that doesn't include the two end values. **Open intervals** are written in the form (x_1, x_2) or $x_1 < x < x_2$. A **closed interval** is a set of numbers between two x-values that does include the two end values. **Closed intervals** are written in the form $[x_1, x_2]$ or $x_1 \le x \le x_2$.

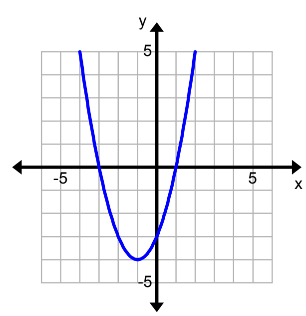
A function *f* is **increasing** when it is rising (or going up) from left to right and it is **decreasing** when it is falling (or going down) from left to right. A **constant** function is neither increasing nor decreasing; it has the same *y*-value for its entire domain.

A function is **positive** when $f(x) > 0$ or the *y*-coordinates are always positive. A function is **negative** when $f(x) < 0$ or the *y*-coordinates are always negative.

Example:

Find the intervals where the function $f(x)=x^2+2x-3$ is:

a. increasing
b. decreasing
c. constant
d. positive
e. negative



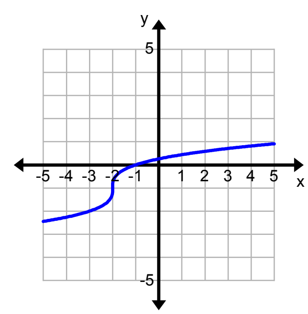
Increasing/Decreasing/Constant	
1. Find the maximums or minimums.	The minimum is (-1, -4).
2. Determine if the function is rising, falling, or constant between the maximums and minimums.	To the left of the minimum the function is falling or decreasing. To the right of the minimum the function is rising or increasing.
3. Write the intervals where the function is increasing, decreasing, or constant using interval notation.	The function is increasing on the interval $(-1,\infty)$. The function is decreasing on the interval $(-\infty,-1)$. The function is never constant.

<table>
<tr><td>Positive/Negative</td><td></td></tr>
<tr><td>1. Find all the x-intercepts of the function.</td><td>The x-intercepts are at (-3, 0) and (1, 0).</td></tr>
<tr><td>2. Determine if the function has positive or negative y-values on the intervals between each x-intercept by testing a point on the interval.</td><td><table>
<tr><td>$x<-3$</td><td>$-3<x<1$</td><td>$x>1$</td></tr>
<tr><td>$x=-4$
$f(-4)=5$
Positive</td><td>$x=0$
$f(0)=-3$
Negative</td><td>$x=2$
$f(2)=5$
Positive</td></tr>
</table></td></tr>
<tr><td>3. Write the intervals where the function is positive or negative using interval notation.</td><td>The function is positive on the intervals $(-\infty,-3)$ and $(1,\infty)$. The function is negative on the interval $(-3,1)$.</td></tr>
</table>

Example:

Find the intervals where the function $f(x)=\sqrt[3]{x+2}-1$ is:

a. increasing
b. decreasing
c. constant
d. positive
e. negative



Increasing/Decreasing/Constant	
1. Find the maximums or minimums.	There are no maximums or minimums.
2. Determine if the function is rising, falling, or constant on its entire domain.	The function is rising from left to right so it is increasing on its entire domain.
3. Write the intervals where the function is increasing, decreasing, or constant using interval notation.	The function is increasing on the interval $(-\infty,\infty)$. The function is never decreasing nor is it constant.

<table>
<tr><td>Positive/Negative
1. Find all the x-intercepts of the function.

2. Determine if the function has positive or negative y-values on the intervals between each x-intercept by testing a point on the interval.

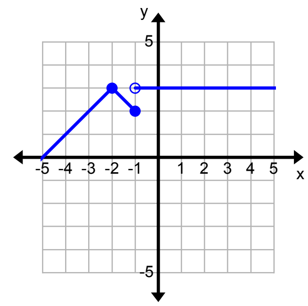
3. Write the intervals where the function is positive or negative using interval notation.</td><td>The x-intercept is (-1, 0).

$x<-1$: $x=-2$, $f(-2)=-1$, Negative
$x>-1$: $x=0$, $f(0)\approx 0.26$, Positive

The function is positive on the interval $(-1,\infty)$. The function is negative on the interval $(-\infty,-1)$.</td></tr>
</table>

Example:

Find the intervals where the function $f\left(x\right)=\begin{cases}-\lvert x+2\rvert+3, & x\le -1\\ 3, & x>-1\end{cases}$ is:



a. increasing
b. decreasing
c. constant
d. positive
e. negative

<table>
<tr><td>Increasing/Decreasing/Constant
1. Find the maximums or minimums and any breaks in the domain.

2. Determine if the function is rising, falling, or constant between each maximum or minimum and each break in the graph.

3. Write the intervals where the function is increasing, decreasing, or constant using interval notation.</td><td>There is a maximum at (-2, 0) and a break in the domain at x = -1.

The function is rising (increasing) to the left of the maximum. It is falling (decreasing) to the right of the maximum. It is constant to the right of x = -1.

The function is increasing on the interval $(-\infty,-2)$. It is decreasing on the interval $(-2,-1)$. It is constant on the interval $(-1,\infty)$.</td></tr>
</table>

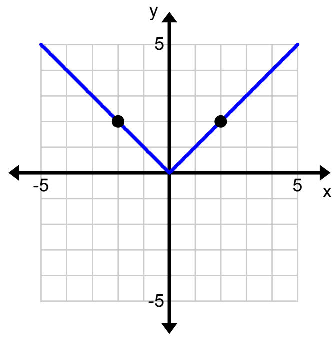
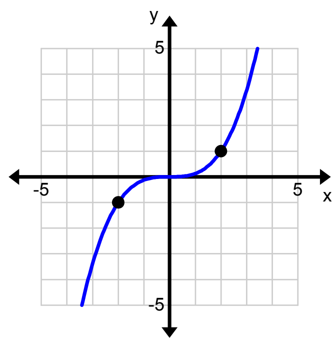
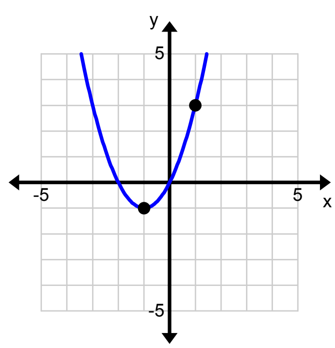
Positive/Negative	
1. Find all the x-intercepts of the function and any places where there is a break in the domain.	The x-intercept is (-5, 0). There is a break in the domain at x = -1.
2. Determine if the function has positive or negative y-values on the intervals between each x-intercept by testing a point on the interval.	<table><tr><td>$x<-5$</td><td>$-5<x<-1$</td><td>$x>-1$</td></tr><tr><td>$x=-6$ $f(-6)=-1$ Negative</td><td>$x=-3$ $f(-3)=2$ Positive</td><td>$x=0$ $f(0)=3$ Positive</td></tr></table>
3. Write the intervals where the function is positive or negative using interval notation.	The function is positive on the intervals $(-5,-1)$ and $(-1,\infty)$. The function is negative on the interval $(-\infty,-5)$.

Practice Exercises C

Find the intervals where the function is:

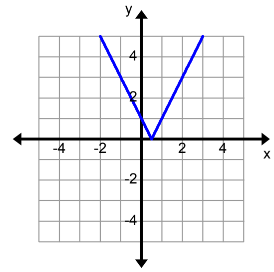
a. increasing
b. decreasing
c. constant
d. positive
e. negative

1. $f(x)=\frac{1}{2}x+3$
2. $f(x)=2x^2+3x-2$
3. $f(x)=2\sqrt{x-3}$
4. $f(x)=\sqrt[3]{x-3}$
5. $f(x)=|x-4|+1$
6. $f(x)=\begin{cases} 2, & x<0 \\ x^2-1, & x>0 \end{cases}$

VOCABULARY	GRAPHICALLY	ALGEBRAICALLY
A function is **symmetric with respect to the *y*-axis** if, for every point (x, y) on the graph, the point $(-x, y)$ is also on the graph. In other words, if you substitute $-x$ in for every x you end up with the original function. When looking at the graph, you could "fold" the graph along the *y*-axis and both sides are the same.		$f(x) = \|x\| + 5$ $f(-x) = \|-x\| + 5$ $f(x) = f(-x) = \|x\| + 5$
A function is **symmetric with respect to the origin** if, for every point (x, y) on the graph, the point $(-x, -y)$ is also on the graph. In other words, if you substitute $-x$ in for every x you end up with the opposite of the original function. When looking at the graph, there is a mirror image in Quadrants 1 & 3 or Quadrants 2 & 4.		$f(x) = 8x^3$ $f(-x) = 8(-x)^3$ $f(-x) = -f(x) = -8x^3$
An equation with **no symmetry**. If you substitute $-x$ in for every x you end up with something that is neither the original function nor its opposite. When looking at the graph, you could not "fold" the graph along the *y*-axis and have both sides are the same or it doesn't reflect a mirror image.		$f(x) = x^2 + 2x$ $f(-x) = (-x)^2 + 2(-x)$ $f(-x) = x^2 - 2x \neq f(x) \neq -f(x)$

Example:

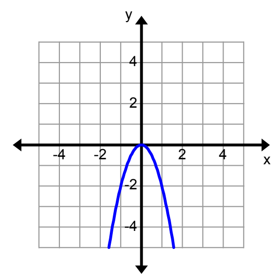
Determine what kind of symmetry, if any, $f\left(x\right)=\left|2x-1\right|$ has.

Test for *y*-axis Symmetry	**Test for Origin Symmetry**	**Graph**
Replace x with $-x$ and see if the result is the same as the original equation. $f\left(-x\right)=\left|2(-x)-1\right|$ $f(-x)=\lvert -2x-1\rvert$ This is not the same as the original equation.	Replace x with $-x$ and see if the result is the opposite of the original equation. $f\left(-x\right)=\left|2(-x)-1\right|$ $f(-x)=\lvert -2x-1\rvert$ This is not the opposite of the original equation.	

The function $f\left(x\right)=\left|2x-1\right|$ has no symmetry.

Example:

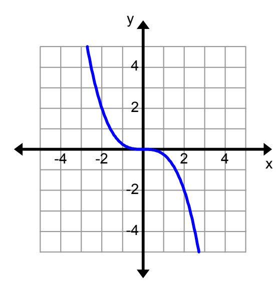
Determine what kind of symmetry, if any, $f\left(x\right)=-2x^2$

Test for *y*-axis Symmetry	**Test for Origin Symmetry**	**Graph**
Replace x with $-x$ and see if the result is the same as the original equation. $f\left(-x\right)=-2(-x)^2$ $f(-x)=-2x^2$ This is equal to the original equation.	Replace x with $-x$ and see if the result is the opposite of the original equation. $f\left(-x\right)=-2(-x)^2$ $f(-x)=-2x^2$ This is not the opposite of the original equation.	

The function $f\left(x\right)=-2x^2$ has y-axis symmetry.

Example:

Determine what kind of symmetry, if any, the function graphed at the right has.



Test for y-axis Symmetry	**Test for origin symmetry**
Pick a point $\left(x,y\right)$ on the graph and see if $\left(-x,y\right)$ is also on the graph. The point (-2, 2) is on the graph but the point (2, 2) is not. The function does not have y-axis symmetry.	Pick a point $\left(x,y\right)$ on the graph and see if $\left(-x,-y\right)$ is also on the graph. The point (-2, 2) is on the graph and the point (2, -2) is also on the graph. The function has origin symmetry.

The function graphed has origin symmetry.

VOCABULARY

End behavior describes what is happening to the *y*-values of a graph when *x* goes to the far right $(+\infty)$ or *x* goes the far left $(-\infty)$.

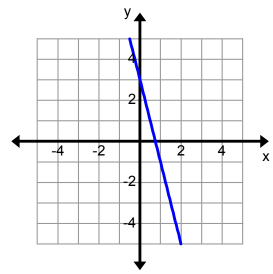
End behavior is written in the following format:

Right End Behavior: $\lim\limits_{x\to\infty} f(x) = c$

Left End Behavior: $\lim\limits_{x\to-\infty} f(x) = c$

Example:

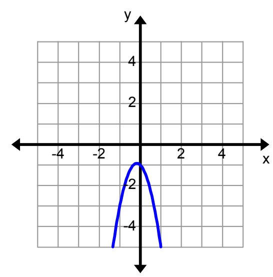
Find the end behavior of $f(x) = -4x+3$.



As *x* gets larger the function is getting more and more negative. Therefore, the right end behavior is $\lim\limits_{x\to\infty} f(x) = -\infty$. As *x* gets smaller the function is getting more and more positive. Therefore the left end behavior is $\lim\limits_{x\to-\infty} f(x) = \infty$.

Example:

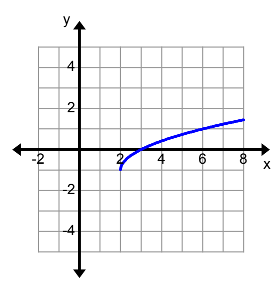
Find the end behavior of $f(x) = -3x^2 - x - 1$.



As *x* gets larger the function is getting more and more negative. Therefore, the right end behavior is $\lim\limits_{x\to\infty} f(x) = -\infty$. As *x* gets smaller the function is getting more and more negative. Therefore the left end behavior is $\lim\limits_{x\to-\infty} f(x) = -\infty$.

Example:

Find the end behavior of $f(x) = \sqrt{x-2} - 1$.



As *x* gets larger the function is getting more and more positive. Therefore, the right end behavior is $\lim\limits_{x\to\infty} f(x) = \infty$. The domain is restricted to numbers greater than or equal to 2, therefore this graph has no left end behavior.

Practice Exercises D

Graph each function below and find the:

a. Domain and Range
b. Intercepts, if any
c. Determine whether the function has any symmetry.
d. List the intervals where the function is increasing, decreasing, or constant.
e. List the intervals where the function is positive or negative.
f. Find all the relative maximums and minimums.
g. Find end behavior

1. $f(x) = 2x - 5$

2. $f(x) = |x-3| + 1$

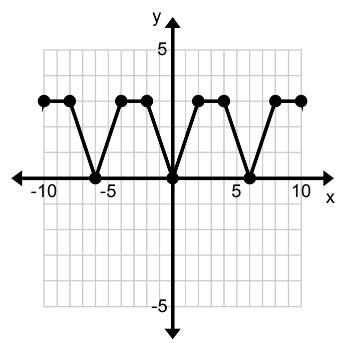
3. $f(x) = \sqrt{x+2} - 4$

4. $f(x) = -(x-3)^2$

5. $f(x) = \sqrt[3]{x-1} + 5$

6. $f(x) = \begin{cases} 3, & x \le -1 \\ \frac{1}{3}x + 2, & -1 < x \le 12 \end{cases}$

VOCABULARY
Periodicity refers to a function with a repeating pattern. The period of this function is 6 horizontal units. Meaning the pattern will repeat itself every 6 horizontal units.

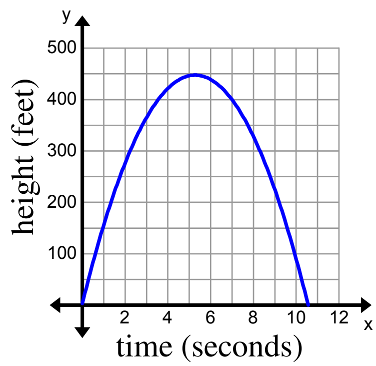


You Decide

Mr. Astro's physics class created rockets for an end of the year competition. There were three groups who constructed rockets. On launch day the following information was presented for review to determine a winner.

Group A estimated that their rocket was easily modeled by the equation: $y = -16x^2 + 176x + 3$.

Group B presented the following graph of the height of their rocket, in feet, over time.



Group C recorded their height in the table below.

Time (seconds)	0	2	4	6	8	10
Height (feet)	3	256	381	377	246	0

Who should be the winner of the competition? Use mathematical reasons to support your conclusion.

Unit 2 Cluster 2 (F.IF.7b)
Graphing Square Root, Cube Root, and Piecewise-Defined Functions, Including Step Functions and Absolute Value Functions

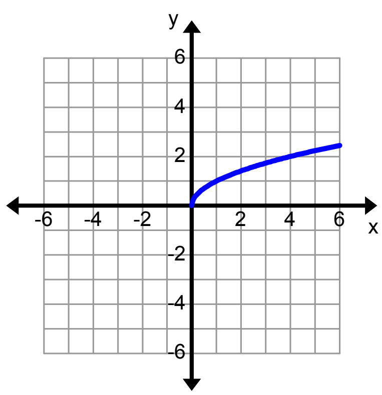
Cluster 2: Analyzing functions using different representations

2.2.1b Graph functions from equations by hand and with technology showing key features (square roots, cube roots, piecewise-defined functions including step functions, and absolute value)

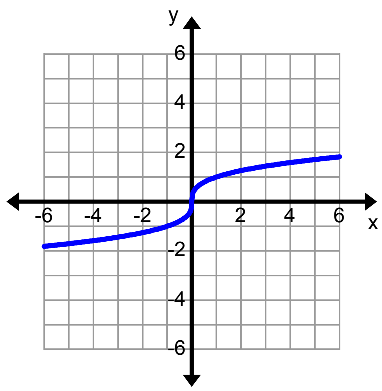
VOCABULARY

There are several types of functions (linear, exponential, quadratic, absolute value, etc.). Each of these could be considered a family with unique characteristics that are shared among the members. The **parent function** is the basic function that is used to create more complicated functions.

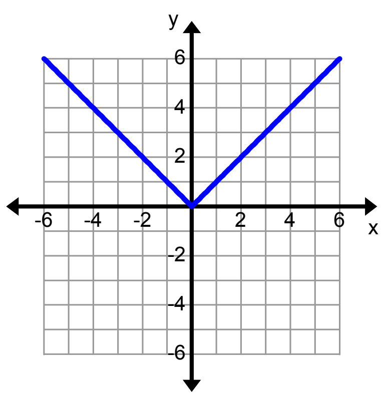
Square Root Function

Parent Function	Key Features
$f(x)=\sqrt{x}=x^{1/2}$ 	**Domain:** $[0,\infty)$ **Range:** $[0,\infty)$ **Intercepts**: x-intercept $(0,0)$, y-intercept $(0,0)$ **Intervals of Increasing/Decreasing:** increasing $(0,\infty)$ **Intervals where Positive/Negative:** $(0,\infty)$ **Relative maximums/minimums**: minimum at $(0,0)$ **Symmetries:** none **End Behavior**: right end behavior $\lim_{x\to\infty}\sqrt{x}=\infty$; left end behavior $\lim_{x\to 0^+}\sqrt{x}=0$

Cube Root Function

Parent Function	Key Features
$f(x)=\sqrt[3]{x}=x^{1/3}$ 	**Domain:** $(-\infty,\infty)$ **Range:** $(-\infty,\infty)$ **Intercepts**: x-intercept $(0,0)$, y-intercept $(0,0)$ **Intervals of Increasing/Decreasing:** increasing $(-\infty,\infty)$ **Intervals where Positive/Negative:** positive $(0,\infty)$, negative $(-\infty,0)$ **Relative maximums/minimums**: none **Symmetries:** origin **End Behavior**: right end behavior $\lim\limits_{x\to\infty}\sqrt[3]{x}=\infty$; left end behavior $\lim\limits_{x\to-\infty}\sqrt[3]{x}=-\infty$

Absolute Value Function

Parent Function	Key Features
$f(x)=\lvert x\rvert$ 	**Domain:** $(-\infty,\infty)$ **Range:** $[0,\infty)$ **Intercepts**: x-intercept $(0,0)$, y-intercept $(0,0)$ **Intervals of Increasing/Decreasing:** increasing $(0,\infty)$, decreasing $(-\infty,0)$ **Intervals where Positive/Negative:** positive $(-\infty,0)\cup(0,\infty)$ **Relative maximums/minimums**: minimum at $(0,0)$ **Symmetries:** y-axis symmetry **End Behavior**: right end behavior $\lim\limits_{x\to\infty}\lvert x\rvert=\infty$; left end behavior $\lim\limits_{x\to-\infty}\lvert x\rvert=\infty$

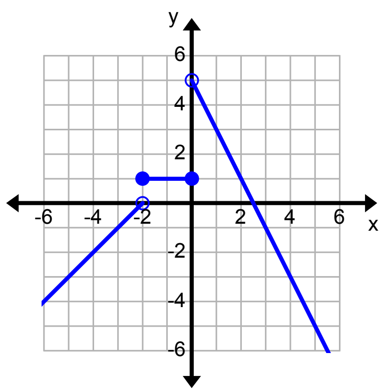
Piecewise-Defined Functions

A piecewise-defined function is a function that consists of pieces of two or more functions. For example $f(x)=\begin{cases} x+2, & x<-2 \\ 1, & -2\le x\le 0 \\ -2x+5, & x>0 \end{cases}$ is a piecewise-defined function. It has a piece of the function $f(x)=x+2$ but only the piece where $x<-2$. It also contains the function $f(x)=1$,

but only where $-2 \le x \le 0$. Finally, it contains the function $f(x) = -2x + 5$ but only where $x > 0$.

Piecewise-Defined Function

$$f(x) = \begin{cases} x+2, & x < -2 \\ 1, & -2 \le x \le 0 \\ -2x+5, & x > 0 \end{cases}$$



Key Features

Domain: $(-\infty, \infty)$

Range: $(-\infty, 5]$

Intercepts: x-intercept $(2.5, 0)$, y-intercept $(0, 1)$

Intervals of Increasing/Decreasing: increasing $(-\infty, 2)$, decreasing $(5, \infty)$

Intervals where Positive/Negative: positive $(-2, 0)$ and $(0, 2.5)$, negative $(-\infty, -2)$ and $(2.5, \infty)$

Relative maximums/minimums: none

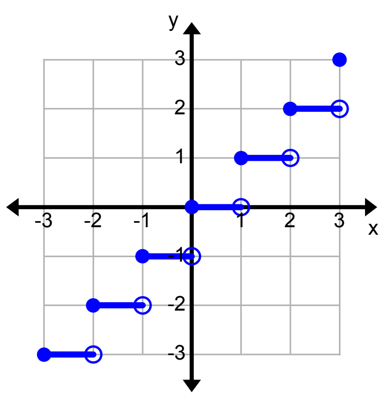
Symmetries: none

End Behavior: right end behavior $\lim_{x\to\infty}(-2x+5) = -\infty$; left end behavior $\lim_{x\to-\infty}(x+2) = -\infty$

Step Functions are piecewise-defined functions made up of constant functions. It is called a step function because the graph resembles a staircase.

Step Function

$f(x) = \text{int}[\![x]\!]$



Key Features

Domain: $(-\infty, \infty)$

Range: $\{y \mid y \text{ is an integer}\}$

Intercepts: x-intercept $x = [0, 1)$ and $y = 0$, y-intercept $(0, 0)$

Intervals of Increasing/Decreasing: neither increasing nor decreasing

Intervals where Positive/Negative: positive $(1, \infty)$, negative $(-\infty, 0)$

Relative maximums/minimums: none

Symmetries: origin

End Behavior: right end behavior $\lim_{x\to\infty} \text{int}[\![x]\!] = \infty$; left end behavior $\lim_{x\to-\infty} \text{int}[\![x]\!] = -\infty$

Unit 2 Cluster 1(F.IF.6) and Cluster 5(F.LE.3)
Quadratic Functions and Modeling

Cluster 1: Interpret Functions that Arise in Applications in Terms of a Context

2.1.3 Average rate of change over an interval: calculate, interpret, and estimate from a graph.

Cluster 5: Constructing and comparing linear, quadratic, and exponential models; solve problems

2.5.1 Exponential functions will eventually outgrow all other functions

VOCABULARY

The **average rate of change** of a function over an interval is the ratio of the difference (change) in *y* over the difference (change) in *x*.

$$\text{average rate of change} = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$$

Example:

Find the average rate of change for $f(x) = 2x^2 - 3x + 1$ on the interval [0, 2].

First, find the value of the function at each end point of the interval.

$f(0) = 2(0)^2 - 3(0) + 1$

$f(0) = 0 - 0 + 1$

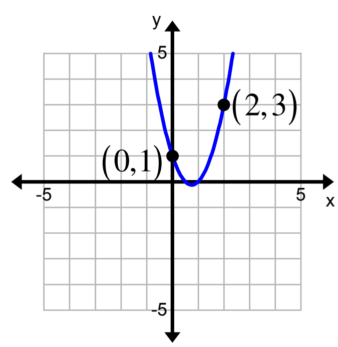
$f(0) = 1$

$f(2) = 2(2)^2 - 3(2) + 1$

$f(2) = 2 \cdot 4 - 6 + 1$

$f(2) = 8 - 6 + 1$

$f(2) = 3$



Next, find the slope between the two points (0, 1) and (2, 3).

$$m = \frac{3-2}{2-0} = \frac{1}{2}$$

The average rate of change of $f(x) = 2x^2 - 3x + 1$ on the interval [0, 2] is $\frac{1}{2}$.

Example:

The per capita consumption of ready-to-eat and ready-to-cook breakfast cereal is shown below. Find the average rate of change from 1992 to 1995 and interpret its meaning.

Years since 1990	0	1	2	3	4	5	6	7	8	9
Cereal Consumption (pounds)	15.4	16.1	16.6	17.3	17.4	17.1	16.6	16.3	15.6	15.5

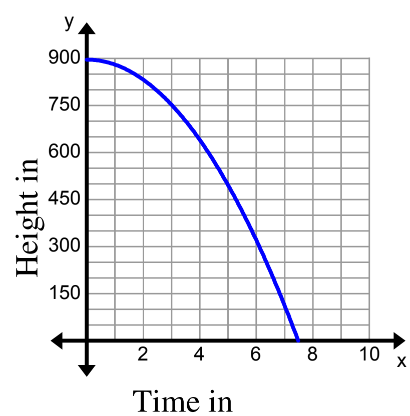
The year 1992 is two years since 1990 and 1995 is 5 years since 1990, therefore the interval is [2, 5]. Find the slope between the two points (2, 16.6) and (5, 17.1).

$$m = \frac{17.1 - 16.6}{5 - 2} = \frac{0.5}{3} = 0.1\overline{6}$$

The average rate of change from 1992 to 1995 is $0.1\overline{6}$ pounds per year. This means that each household increased their cereal consumption an average of $0.1\overline{6}$ pounds each year from 1992 to 1995.

Example:

Joe is visiting the Eiffel Tower in Paris. He has always enjoyed mathematics and uses his visit as an opportunity to test how long it will take for a water balloon to hit the ground from the top floor. The balloon's height is graphed. Use the graph to estimate the average rate of change of the balloon from 4 to 7 seconds and interpret its meaning.



At 4 seconds the height of the water balloon is approximately 650 feet. At 7 seconds the height of the balloon is approximately 100 feet. Find the slope between the points (4, 650) and (7, 100).

$$m = \frac{100 - 650}{7 - 4} = \frac{-550}{3} = -183.\overline{3}$$

The negative indicates the balloon is falling. The balloon is picking up speed as it is falling. This means that for each second the balloon is falling from 4 to 7 seconds, it increases in speed an average of 183.3 feet per second from 4 to 7 seconds.

Practice Exercises A

Find the average rate of change for each function on the specified interval.

1. $f(x) = 3x^2 - x + 5$ on [-1, 3]
2. $f(x) = 4x^2 + 12x + 9$ on [-3, 0]
3. $f(x) = -x^2 + 4$ on [-4, -2]
4. $f(x) = -2x^2 - 6x$ on [-1, 0]

Find the average rate of change on the specified interval and interpret its meaning.

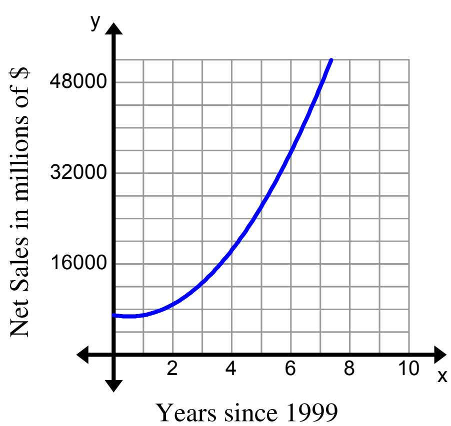
5. Many of the elderly are placed in nursing care facilities. The cost of these has risen significantly since 1960. Use the table below find the average rate of change from 2000 to 2010.

Years since 1960	Nursing Care Cost (billions of $)
0	1
10	4
20	18
30	53
40	96
50	157

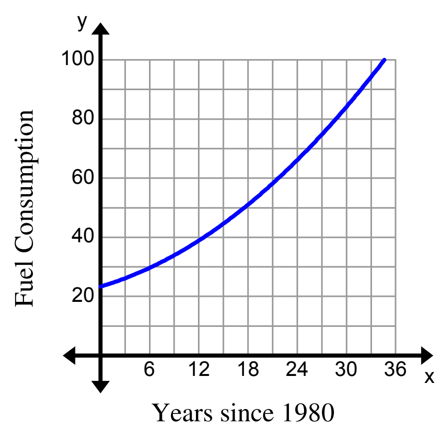
6. The height of an object thrown straight up is shown in the table below. Find the average rate of change from 1 to 2 seconds.

Time (seconds)	Height (feet)
0	140
1	162
2	152
3	110
4	36

7. The net sales of a company are shown in the graph below. Estimate the average rate of change for 2007 to 2009.



8. The graph below shows fuel consumption in billions of gallons for vans, pickups and SUVs. Estimate the average rate of change for 2005 to 2012.



Practice Exercises B

Complete the tables.

x	$f(x) = 2x$
-2	
-1	
0	
1	
2	
3	
4	
5	

x	$g(x) = x^2$
-2	
-1	
0	
1	
2	
3	
4	
5	

x	$h(x) = 2^x$
-2	
-1	
0	
1	
2	
3	
4	
5	

Practice Exercises C

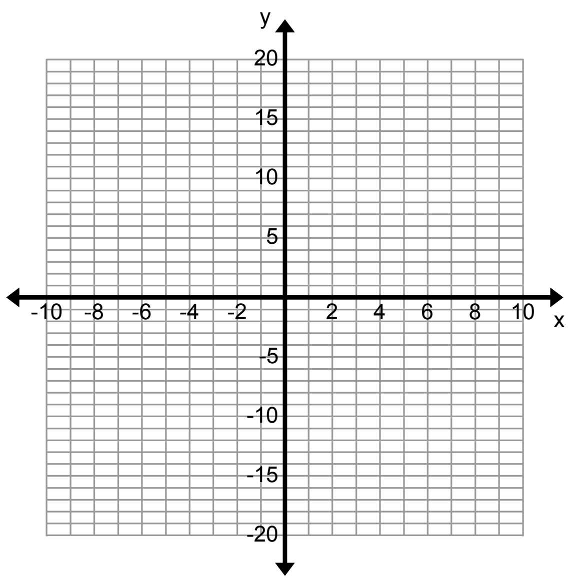
Find the average rate of change for functions $f(x), g(x), and\ h(x)$ for the specified intervals. Determine which of the three functions is increasing the fastest.

1. [0, 1]
2. [3, 5]
3. [-2, 5]
4. [0, 3]
5. [-2, 0]
6. [0, 5]

Practice Exercises D

1. Graph the following functions on the same coordinate plane.

a. $k(x)=\frac{3}{2}x-1$ b. $p(x)=2(x-3)^2-5$ c. $r(x)=3^{x-2}-7$



2. Find the average rate of change for functions $f(x), g(x), and\ h(x)$ for the specified intervals. Determine which of the three functions is increasing the fastest.

a. [-4, 2] b. [3, 5] c. [0, 10]

You Decide

Use exercises C and D to help you answer the following questions.

1. For each exercise, determine which function has the greatest average rate of change on the interval $[0,\infty)$?

2. In general, what type of function will increase faster? Explain your reasoning.

FACTORING
(To be used before F.IF.8)

VOCABULARY
Factoring is the reverse of multiplication. It means to write an equivalent expression that is a product. Each of the items that are multiplied together in a product is a **factor**. An expression is said to be **factored completely** when all of the factors are **prime polynomials,** that is they cannot be factored any further. The **greatest common factor** is the largest expression that all the terms have in common.

FACTOR OUT A COMMON TERM

Example: $2x^2+6x+8$	What is the largest factor that evenly divides $2x^2, 6x$, and 8 ?
$2x^2 : 1 \cdot 2 \cdot x \cdot x$ $6x : 1 \cdot 2 \cdot 3 \cdot x$ $8 : 1 \cdot 2 \cdot 2 \cdot 2$	The common numbers are 1and 2. Multiply them and the product is the greatest common factor.
$\frac{2x^2}{2}+\frac{6x}{2}+\frac{8}{2}$	Divide each term by the greatest common factor.
$2\left(x^2+3x+4\right)$	Rewrite with the common term on the outside of the parenthesis and the simplified terms inside the parenthesis.

Example: $8w^4-3w^3+5w^2$	What is the largest factor that evenly divides $8w^4, 3w^3,$ and $5w^2$?
$8w^4 : w \cdot w \cdot w \cdot w \cdot 1 \cdot 2 \cdot 2 \cdot 2$ $-3w^3 : w \cdot w \cdot w \cdot -1 \cdot 3$ $5w^2 : w \cdot w \cdot 1 \cdot 5$	The common numbers are w and w. Multiply them and the product is the greatest common factor.
$\frac{8w^4}{w^2}-\frac{3w^3}{w^2}+\frac{5w^2}{w^2}$	Divide each term by the greatest common factor.
$w^2\left(8w^2-3w+5\right)$	Rewrite with the common term on the outside of the parenthesis and the simplified terms inside the parenthesis.

Example: $$9z^3 - 3z^2 + 15z$$	What is the largest factor that evenly divides $9z^3, 12z^2,$ and $15z$?
$9z^3 : 1 \cdot 3 \cdot 3 \cdot z \cdot z \cdot z$ $-3z^2 : -1 \cdot 3 \cdot z \cdot z$ $15z : 1 \cdot 3 \cdot 5 \cdot z$	The common numbers are 3 and z. Multiply them and the product is the greatest common factor.
$\dfrac{9z^3}{3z} - \dfrac{3z^2}{3z} + \dfrac{15z}{3z}$	Divide each term by the greatest common factor.
$3z\left(3z^2 - z + 5\right)$	Rewrite with the common term on the outside of the parenthesis and the simplified terms inside the parenthesis.

Practice Exercises A

Factor out the greatest common factor.

1. $4x^2 - 12x - 16$
2. $5x^5 + 10x^4 + 15x^3$
3. $-8x^4 - 32x^3 + 16x^2$
4. $-2x^3 - x^2 - 3x$
5. $27x^2 + 36x - 18$
6. $14x^2 - 21x + 49$

FACTOR A TRINOMIAL WITH A LEADING COEFFICIENT OF 1

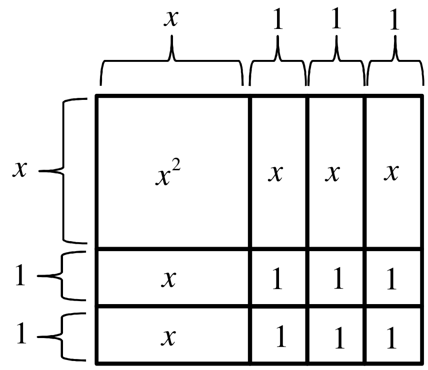
- When factoring a trinomial of the form $ax^2 + bx + c$,where a = 1, and b and c are integers, find factors of c that add to equal b.

  If p and q are the factors, the factored form looks like $(x + p)(x + q)$.

Example: $$x^2 + 5x + 6$$	
(see table below)	Find factors of 6 that add to be 5. The factors are 2 and 3.
$(x+2)(x+3)$	This is the factored form.

Factors of 6		**Sum (adds to be)**
1	6	7
2	3	5
-1	-6	-7
-2	-3	-5

Another way to look at factoring is with an area model like the one pictured below.



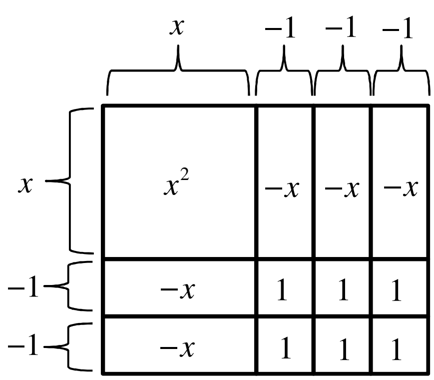
The rectangular area represents the trinomial x^2+5x+6. The width across the top is $x+1+1+1$ or $x+3$ and the length down the side is $x+1+1$ or $x+2$. To obtain the area of the rectangle, you would multiply the length times the width or $(x+2)(x+3)$. Notice that this result is the same as when we found factors of the constant term that added to the coefficient of the *x* term.

- When factoring a trinomial of the form ax^2-bx+c,where *a* = 1, and *b* and *c* are integers, find factors of *c* that add to equal *b*.

  If *p* and *q* are the factors, the factored form looks like $(x-p)(x-q)$.

<table>
<tr><td>Example:
x^2-5x+6</td><td></td></tr>
<tr><td>
<table>
<tr><th colspan="2">Factors of 6</th><th>Sum
(adds to be)</th></tr>
<tr><td>1</td><td>6</td><td>7</td></tr>
<tr><td>2</td><td>3</td><td>5</td></tr>
<tr><td>-1</td><td>-6</td><td>-7</td></tr>
<tr><td>-2</td><td>-3</td><td>-5</td></tr>
</table>
</td><td>Find factors of 6 that add to be -5. The factors are -2 and -3.</td></tr>
<tr><td>$(x-2)(x-3)$</td><td>This is the factored form.</td></tr>
</table>

This example can also be modeled with an area model as the picture below demonstrates.



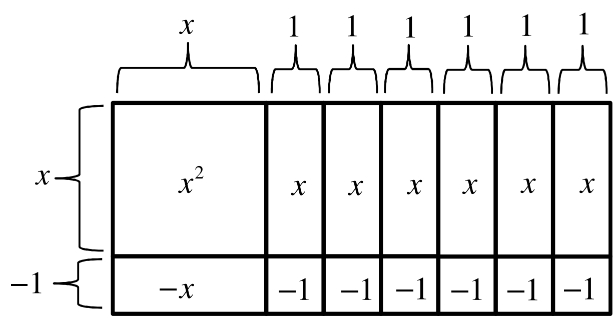
The rectangular area represents the trinomial x^2-5x+6. The width across the top is $x-1-1-1$ or $x-3$ and the length down the side is $x-1-1$ or $x-2$. To obtain the area of the rectangle, you would multiply the length times the width or $(x-2)(x-3)$. Notice that this result is the same as when we found factors of the constant term that added to the coefficient of the *x* term.

- When factoring a trinomial of the form ax^2+bx-c ,where a = 1, and b and c are integers, find factors of c that add to equal b.

  If p and q are the factors, the factored form looks like $(x+p)(x-q)$.

<table>
<tr><td>Example:
x^2+5x-6</td><td></td></tr>
<tr><td>
<table>
<tr><th colspan="2">Factors of 6</th><th>Sum (adds to be)</th></tr>
<tr><td>1</td><td>-6</td><td>-5</td></tr>
<tr><td>2</td><td>-3</td><td>-1</td></tr>
<tr><td>-1</td><td>6</td><td>5</td></tr>
<tr><td>-2</td><td>3</td><td>1</td></tr>
</table>
</td><td>Find factors of -6 that add to be 5. The factors are -1 and 6.</td></tr>
<tr><td>$(x-1)(x+6)$</td><td>This is the factored form.</td></tr>
</table>

This example can also be modeled with an area model as the picture below demonstrates.



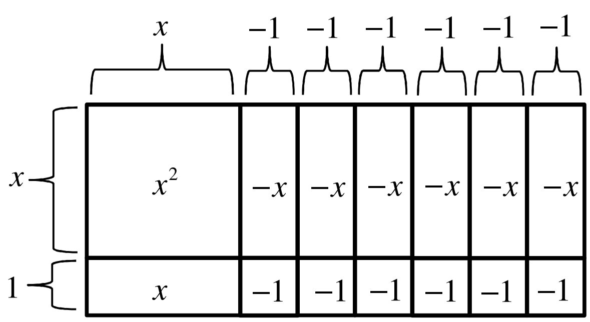
The rectangular area represents the trinomial x^2+5x-6. The width across the top is $x+1+1+1+1+1+1$ or $x+6$ and the length down the side is $x-1$. To obtain the area of the rectangle, you would multiply the length times the width or $(x-1)(x+6)$. Notice that this result is the same as when we found factors of the constant term that added to the coefficient of the x term.

- When factoring a trinomial of the form ax^2-bx-c ,where a = 1, and b and c are integers, find factors of c that add to equal b.

  If p and q are the factors, the factored form looks like $(x+p)(x-q)$.

<table>
<tr><td>Example:
$x^2 - 5x - 6$</td><td></td></tr>
<tr><td>
<table>
<tr><th colspan="2">Factors of 6</th><th>Sum (adds to be)</th></tr>
<tr><td>1</td><td>-6</td><td>-5</td></tr>
<tr><td>2</td><td>-3</td><td>-1</td></tr>
<tr><td>-1</td><td>6</td><td>5</td></tr>
<tr><td>-2</td><td>3</td><td>1</td></tr>
</table>
</td><td>Find factors of -6 that add to be -5. The factors are 1 and -6.</td></tr>
<tr><td>$(x+1)(x-6)$</td><td>This is the factored form.</td></tr>
</table>

This example can also be modeled with an area model as the picture below demonstrates.



The rectangular area represents the trinomial $x^2 - 5x - 6$. The width across the top is $x-1-1-1-1-1-1$ or $x-6$ and the length down the side is $x+1$. To obtain the area of the rectangle, you would multiply the length times the width or $(x+1)(x-6)$. Notice that this result is the same as when we found factors of the constant term that added to the coefficient of the *x* term.

Practice Exercises B

Factor each expression.

1. $x^2 - 4x - 21$
2. $x^2 + 4x - 12$
3. $x^2 - x - 2$
4. $x^2 - 9x + 18$
5. $x^2 + 4x - 12$
6. $x^2 - 5x - 36$
7. $x^2 + 15xy + 14y^2$
8. $x^2 + 3xy + 2y^2$
9. $x^2 - 17xy + 72y^2$

VOCABULARY
A **perfect square** is a number that can be expressed as the product of two equal integers. For example: 100 is a perfect square because $10 \cdot 10 = 100$ and x^2 is a perfect square because $x \cdot x = x^2$.

FACTOR USING THE DIFFERENCE OF TWO SQUARES

When something is in the form $a^2 - b^2$, where *a* and *b* are perfect square expressions, the factored form looks like $(a-b)(a+b)$.

Example: $x^2 - 49$	
$x^2 = x \cdot x$ $49 = 7 \cdot 7$	x^2 and 49 are both perfect squares and you are finding the difference between them, so you can use the difference of two squares to factor. Therefore: $a = x$ and $b = 7$
$(x-7)(x+7)$	This is the factored form.

Example: $25x^2 - 36y^2$	
$25x^2 = 5x \cdot 5x$ $36y^2 = 6y \cdot 6y$	$25x^2$ and $36y^2$ are both perfect squares and you are finding the difference between them, so you can use the difference of two squares to factor. Therefore: $a = 5x$ and $b = 6y$
$(5x-6y)(5x+6y)$	This is the factored form.

Practice Exercises C

Factor each expression.

1. $49x^2 - 25$	3. $9x^2 - 4$	5. $36x^2 - 121$
2. $x^2 - 64y^2$	4. $16x^2 - 81y^2$	6. $100x^2 - 64y^2$

FACTOR BY GROUPING

When factoring a trinomial of the form $ax^2 + bx + c$,where *a, b*, and *c* are integers, you will need to use the technique of factoring by grouping.

Example: $6x^2 - x - 15$	
$(6)(15) = 90$	Multiply the leading coefficient and the constant.
Factors of 90 1, 90 2, 45 3, 30 5, 18 6, 15 9, 10	Choose the combination that will either give the sum or difference needed to result in the coefficient of the *x* term. In this case the difference should be -1, so 9 and -10 will give you the desired result. Or in other words, when you combine $9x$ and $-10x$ you will end up with $-x$.
$6x^2 + 9x - 10x - 15$	Rewrite the equation using the combination in place of the middle term.
$(6x^2 + 9x) + (-10x - 15)$	Group the first two terms and the last two terms together in order to factor.
$3x(2x+3) - 5(2x+3)$	Factor the greatest common factor out of each group.
$(2x+3)$	Write down what is in the parenthesis (they should be identical). This is one of the factors.
$(2x+3)(3x-5)$	Add the "left-overs" to obtain the second factor.

<table>
<tr><td>Example:
$12x^2 + 7x - 10$</td><td></td></tr>
<tr><td>(12)(10) = 120</td><td>Multiply the leading coefficient and the constant.</td></tr>
<tr><td><table><tr><th colspan="2">Factors of 90</th></tr><tr><td>1</td><td>120</td></tr><tr><td>2</td><td>60</td></tr><tr><td>3</td><td>40</td></tr><tr><td>4</td><td>30</td></tr><tr><td>5</td><td>24</td></tr><tr><td>6</td><td>20</td></tr><tr><td>8</td><td>15</td></tr><tr><td>10</td><td>12</td></tr></table></td><td>Choose the combination that will either give the sum or difference needed to result in the coefficient of the x term.

In this case the difference should be 7, so -8 and 15 will give you the desired result. Or in other words, when you combine $-8x$ and $15x$ you will end up with $7x$.</td></tr>
<tr><td>$12x^2 - 8x + 15x - 10$</td><td>Rewrite the equation using the combination in place of the middle term.</td></tr>
<tr><td>$\left(12x^2 - 8x\right) + \left(15x - 10\right)$</td><td>Group the first two terms and the last two terms together in order to factor.</td></tr>
<tr><td>$4x\left(3x - 2\right) + 5\left(3x - 2\right)$</td><td>Factor the greatest common factor out of each group.</td></tr>
<tr><td>$\left(3x - 2\right)$</td><td>Write down what is in the parenthesis (they should be identical). This is one of the factors.</td></tr>
<tr><td>$\left(3x - 2\right)\left(4x + 5\right)$</td><td>Add the “left-overs” to obtain the second factor.</td></tr>
</table>

<table>
<tr><td>Example:
$4x^2 - 25$</td><td></td></tr>
<tr><td>(4)(25) = 100</td><td>Multiply the leading coefficient and the constant.</td></tr>
<tr><td><table><tr><th colspan="2">Factors of 100</th></tr><tr><td>1</td><td>100</td></tr><tr><td>2</td><td>50</td></tr><tr><td>4</td><td>25</td></tr><tr><td>5</td><td>20</td></tr><tr><td>10</td><td>10</td></tr></table></td><td>Choose the combination that will either give the sum or difference needed to result in the coefficient of the x term.

In this case the difference should be 0, so -10 and 10 will give you the desired result. Or in other words, when you combine $-10x$ and $10x$ you will end up with 0.</td></tr>
</table>

$4x^2-10x+10x-25$	Rewrite the equation using the combination in place of the middle term.
$\left(4x^2-10x\right)+\left(10x-25\right)$	Group the first two terms and the last two terms together in order to factor.
$2x\left(2x-5\right)+5\left(2x-5\right)$	Factor the greatest common factor out of each group.
$\left(2x-5\right)$	Write down what is in the parenthesis (they should be identical). This is one of the factors.
$\left(2x-5\right)\left(2x+5\right)$	Add the "left-overs" to obtain the second factor.

Practice Exercises D
Factor the expression.

1. $2x^2+13x+6$
2. $4x^2+3x+1$
3. $3x^2+2x-8$
4. $2x^2-11x-6$
5. $2x^2+4x+2$
6. $3x^2-6x+3$
7. $10x^2-x-6$
8. $6x^2-7x-20$
9. $12x^2+17x+6$

FACTORING GUIDELINES

#1: Always look for a greatest common factor. Then factor it out if there is one.
#2: Count the number of terms. If there are two terms, determine if you can use the difference of two squares. If you can, factor. If not, proceed to #3.
#3: If there are three terms, check the leading coefficient. If it is "1", then find factors of the constant term that add to the coefficient of the *x*-term. If not, proceed to #4.
#4: If the leading coefficient is not "1", factor by grouping.

Mixed practices E
Factor the expression.

1. $2x^2-50$
2. $-2x^2+16xy-32y^2$
3. $3x^2-5x-12$
4. $5x^2+10x+5$
5. $25x^2-64y^2$
6. $3x^2-27$
7. $4y^2+4y$
8. $x^2+13x+42$
9. $4x^2+12x+9$
10. x^2-6x
11. $9x^2-12x+4$
12. $8x^2+2x-3$

Unit 2 Cluster 2 (F.IF.8) , Unit 3 Cluster 1 (A.SSE.1a) and Unit 3 Cluster 2 (A.SSE.3a,b)
Forms of Quadratic Functions

Cluster 2: Analyzing functions using different representations

2.2.2 Writing functions in different but equivalent forms (quadratics: standard, vertex, factored) using the processes of factoring or completing the square to reveal and explain different properties of functions. Interpret these in terms of a context.

Cluster 1: Interpret the structure of expressions

3.1.1a Interpret parts of an expression, such as terms, factors, and coefficients

Cluster 2: Writing expressions in equivalent forms and solving

3.2.1 Choose an appropriate from of an equation to solve problems (factor to find zeros, complete the square to find maximums and minimums

VOCABULARY

Forms of **Quadratic Functions**

Standard Form: $f(x) = ax^2 + bx + c$, where $a \neq 0$. Example: $f(x) = 4x^2 - 6x + 3$

Vertex Form: $f(x) = a(x-h)^2 + k$, where $a \neq 0$. Example: $f(x) = 2(x+3)^2 + 5$

Factored Form: $f(x) = a(x-p)(x-q)$, where $a \neq 0$. Example: $f(x) = (x-4)(x+7)$

A **zero of a function** is a value of the input x that makes the output $f(x)$ equal zero. The zeros of a function are also known as roots, x-intercepts, and solutions of $ax^2 + bx + c = 0$.

The **Zero Product Property** states that if the product of two quantities equals zero, at least one of the quantities equals zero. If $ab = 0$ then $a = 0$ or $b = 0$.

Finding Zeros (Intercepts) of a Quadratic Function

When a function is in factored form, the Zero Product Property can be used to find the zeros of the function.

If $f(x) = ax(x-p)$ then $ax(x-p) = 0$ can be used to find the zeros of $f(x)$.

If $0 = ax(x-p)$ then either $ax = 0$ or $(x-p) = 0$.

Therefore, either $x = 0$ or $x = p$.

Example: Find the zeros of $f(x)=2x(x+7)$

$f(x)=2x(x+7)$	
$2x(x+7)=0$	Substitute zero in for $f(x)$.
$2x=0$ or $x+7=0$	Use the zero product property to set each factor equal to zero.
$x=0$ or $x=-7$	Solve each equation.
The zeros are $(0,0)$ and $(-7,0)$	Write them as ordered pairs.

If $f(x)=(x-p)(x-q)$ then $(x-p)(x-q)=0$ can be used to find the zeros of $f(x)$.
If $(x-p)(x-q)=0$ then either $(x-p)=0$ or $(x-q)=0$.
Therefore, either $x=p$ or $x=q$.

Example: Find the zeros of $f(x)=(x-5)(x+9)$

$f(x)=(x-5)(x+9)$	
$(x-5)(x+9)=0$	Substitute zero in for $f(x)$.
$x-5=0$ or $x+9=0$	Use the zero product property to set each factor equal to zero.
$x=5$ or $x=-9$	Solve each equation.
The zeros are $(5,0)$ and $(-9,0)$	Write them as ordered pairs.

NOTE: If a quadratic function is given in standard form, factor first then apply the Zero Product Property.

Example: Find the zeros of $f(x)=x^2-11x+24$

$f(x)=x^2-11x+24$	
$x^2-11x+24=0$	Substitute zero in for $f(x)$.
$(x-8)(x-3)=0$	Factor the trinomial. (See factoring lesson in Unit 2 for extra help.)
$x-8=0$ or $x-3=0$	Use the zero product property to set each factor equal to zero.
$x=8$ or $x=3$	Solve each equation.
The zeros are $(8,0)$ and $(3,0)$	Write them as ordered pairs.

Example: Find the zeros of $f(x) = 4x^2 - 4x - 15$

$f(x) = 4x^2 - 4x - 15$	
$4x^2 - 4x - 15 = 0$	Substitute zero in for *f*(*x*).
$(2x-5)(2x+3) = 0$	Factor the trinomial.
$2x - 5 = 0$ or $2x + 3 = 0$	Use the zero product property to set each factor equal to zero.
$2x = 5 \quad 2x = -3$	
$x = \frac{5}{2}$ or $x = -\frac{3}{2}$	Solve each equation.
The zeros are $\left(\frac{5}{2}, 0\right)$ and $\left(-\frac{3}{2}, 0\right)$	Write them as ordered pairs.

Practice Exercises A

Find the zeros of each function.

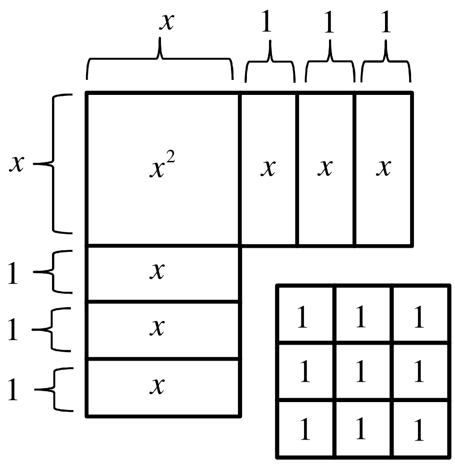
1. $f(x) = -x(x+7)$
2. $f(x) = 2x(x-6)$
3. $f(x) = (x+13)(x-4)$
4. $f(x) = (x-21)(x-3)$
5. $f(x) = x^2 - 7x + 6$
6. $f(x) = x^2 - x + 2$
7. $f(x) = x^2 + 8x + 12$
8. $f(x) = x^2 + 10x - 24$
9. $f(x) = 4x^2 - 12x$
10. $f(x) = 9x^2 - 25$

8. $f(x) = 5x^2 - 3x - 20$

12. $f(x) = 3x^2 + 17x + 10$

COMPLETING THE SQUARE		
To complete the square of $x^2 \pm bx$, add $\left(\frac{b}{2}\right)^2$. In other words, divide the x coefficient by two and square the result.	$x^2 + bx + \underline{\quad}$ $x^2 + bx + \underline{\left(\frac{b}{2}\right)^2}$ $\left(x + \frac{b}{2}\right)\left(x + \frac{b}{2}\right)$ $\left(x + \frac{b}{2}\right)^2$	$x^2 + 6x + \underline{\quad}$ $x^2 + 6x + \underline{\left(\frac{6}{2}\right)^2}$ $x^2 + 6x + \underline{3^2}$ $x^2 + 6x + \underline{9}$ $(x+3)(x+3)$ $(x+3)^2$

An area model can be used to represent the process of completing the square for the expression $x^2+6x+\_\_\_$.



The goal is to arrange the pieces into a square. The x pieces are divided evenly between the two sides so that each side is $(x+3)$ long. However, there is a large piece of the square that is missing. In order to complete the square you need to add 9 ones pieces.

To complete the square of x^2+bx, add $\left(\frac{b}{2}\right)^2$. In other words, divide the x coefficient by two and square the result.	$x^2+bx+\underline{\quad}$ $x^2+bx+\underline{\left(\frac{b}{2}\right)^2}$ $\left(x+\frac{b}{2}\right)\left(x+\frac{b}{2}\right)$ $\left(x+\frac{b}{2}\right)^2$	$x^2+5x+\underline{\quad}$ $x^2+5x+\underline{\left(\frac{5}{2}\right)^2}$ $x^2+5x+\underline{\left(\frac{25}{4}\right)}$ $\left(x+\frac{5}{2}\right)\left(x+\frac{5}{2}\right)$ $\left(x+\frac{5}{2}\right)^2$
To complete the square of ax^2+bx, factor out the leading coefficient, a, giving you $a\left(x^2+\frac{b}{a}x\right)$. Now add $\left(\frac{b}{2a}\right)^2$, which is the square of the coefficient of x divided by two.	$ax^2+bx+\underline{\quad}$ $a\left(x^2+\frac{b}{a}x+\underline{\quad}\right)$ $a\left(x^2+\frac{b}{a}x+\underline{\left(\frac{b}{2a}\right)^2}\right)$ $a\left(x+\frac{b}{2a}\right)\left(x+\frac{b}{2a}\right)$ $a\left(x+\frac{b}{2a}\right)^2$	$3x^2-6x+\underline{\quad}$ $3\left(x^2-\frac{6}{3}x+\underline{\quad}\right)$ $3\left(x^2-\frac{6}{3}x+\underline{\left(\frac{6}{2\cdot 3}\right)^2}\right)$ $3\left(x^2-2x+\underline{(1)^2}\right)$ $3(x-1)(x-1)$ $3(x-1)^2$

Practice Exercises B

For each expression complete the square.

1. $x^2 + 10x +$ ___
2. $x^2 - 7x +$ ___
3. $x^2 - 22x +$ ___
4. $4x^2 - 16x +$ ___
5. $2x^2 + 12x +$ ___
6. $5x^2 + 20x +$ ___

Finding Maximum/Minimum (the vertex) Points of a Quadratic Function

VOCABULARY

Remember when a quadratic function is in vertex form $f(x) = a(x-h)^2 + k$ the point (h,k) is the vertex of the parabola. The value of a determines whether the parabola opens up or down.

The vertex of a parabola that opens up, when $a > 0$, is the **minimum point** of a quadratic function.

The vertex of a parabola that opens up, when $a < 0$, is the **maximum point** of a quadratic function.

Example:

Find the vertex of $f(x) = (x+2)^2 - 3$, then determine whether it is a maximum or minimum point.

$f(x) = (x+2)^2 - 3$ $f(x) = (x-(-2))^2 + (-3)$	Rewrite the equation so it is in the general vertex form $f(x) = a(x-h)^2 + k$.
Vertex: (-2, -3)	$h = -2$ and $k = -3$
The vertex is a minimum.	The leading coefficient is 1, which makes $a > 0$

Example:

Find the vertex of $f(x) = -5(x-8)^2 + 4$, then determine whether it is a maximum or minimum point.

$f(x) = -5(x-8)^2 + 4$	
$f(x) = -5(x-8)^2 + 4$	This equation is already in the general vertex form $f(x) = a(x-h)^2 + k$.
Vertex: (8, 4)	$h = 8$ and $k = 4$
The vertex is a maximum.	The leading coefficient is -5, which makes $a < 0$.

Practice Exercises C

Find the vertex and determine whether it is a maximum or minimum point.

1. $f(x) = 4(x-5)^2 - 3$
2. $f(x) = -(x+3)^2 + 7$
3. $f(x) = 6x^2 + 5$
4. $f(x) = -2(x+6)^2$
5. $f(x) = -5(x-2)^2 + 3$
6. $f(x) = 7(x-1)^2 - 2$

NOTE: If a quadratic function is given in standard form, complete the square to rewrite the equation in vertex form.

Example:

Find the vertex of $f(x) = x^2 + 12x + 7$, then determine whether it is a maximum or minimum point.

$f(x) = x^2 + 12x + 7$	
$f(x) = \left(x^2 + 12x + \underline{\quad}\right) + 7$	Collect variable terms together inside parenthesis with constant term outside the parenthesis.
$f(x) = \left(x^2 + 12x + \left(\frac{12}{2}\right)^2\right) + 7 - \left(\frac{12}{2}\right)^2$ $f(x) = \left(x^2 + 12x + (6)^2\right) + 7 - (6)^2$	Complete the square by adding $\left(\frac{b}{2}\right)^2$ inside the parenthesis. Now subtract $\left(\frac{b}{2}\right)^2$ outside the parenthesis to maintain equality. In other words you are really adding zero to the equation.

$f(x)=\left(x^2+12x+36\right)+7-36$	Simplify
$f(x)=\left(x+6\right)^2-29$	Factor and combine like terms.
Vertex: (-6, -29)	$h=-6$ and $k=-29$
The vertex is a minimum.	The leading coefficient is 1, which makes $a>0$

Example: Find the vertex of $f(x)=3x^2+18x-2$, then determine whether it is a maximum or minimum point.

$f(x)=3x^2+18x-2$	
$f(x)=\left(3x^2+18x+\underline{\quad}\right)-2$	Collect variable terms together inside parenthesis with constant term outside the parenthesis.
$f(x)=3\left(x^2+6x+\underline{\quad}\right)-2$	Factor out the leading coefficient. In this case 3.
$f(x)=3\left(x^2+6x+\left(\frac{6}{2}\right)^2\right)-2-3\cdot\left(\frac{6}{2}\right)^2$ $f(x)=3\left(x^2+6x+(3)^2\right)-2-3\cdot(3)^2$	Complete the square by adding $\left(\frac{b}{2}\right)^2$ inside the parenthesis. Notice that everything in the parenthesis is multiplied by 3 so we need to subtract $3\cdot\left(\frac{b}{2}\right)^2$ outside the parenthesis to maintain equality. In other words you are really adding zero to the equation.
$f(x)=3\left(x^2+6x+9\right)-2-27$	Simplify
$f(x)=3\left(x+3\right)^2-29$	Factor and combine like terms.
Vertex: (-3, -29)	$h=-3$ and $k=-29$
The vertex is a minimum.	The leading coefficient is 3, which makes $a>0$

Example: Find the vertex of $f(x) = -4x^2 - 8x + 3$, then determine whether it is a maximum or minimum point.

$f(x) = -4x^2 - 8x + 3$	
$f(x) = \left(-4x^2 - 8x + \underline{\quad}\right) + 3$	Collect variable terms together inside parenthesis with constant term outside the parenthesis.
$f(x) = -4\left(x^2 + 2x + \underline{\quad}\right) + 3$	Factor out the leading coefficient. In this case -4.
$f(x) = -4\left(x^2 + 2x + \left(\frac{2}{2}\right)^2\right) + 3 - (-4)\cdot\left(\frac{2}{2}\right)^2$ $f(x) = -4\left(x^2 + 2x + (1)^2\right) + 3 - (-4)\cdot(1)^2$	Complete the square by adding $\left(\frac{b}{2}\right)^2$ inside the parenthesis. Notice that everything in the parenthesis is multiplied by -4 so we need to subtract $-4\cdot\left(\frac{b}{2}\right)^2$ outside the parenthesis to maintain equality. In other words you are really adding zero to the equation.
$f(x) = -4\left(x^2 + 2x + 1\right) + 3 + 4$	Simplify
$f(x) = -4(x+1)^2 + 7$	Factor and combine like terms.
Vertex: (-1, 7)	$h = -1$ and $k = 7$
The vertex is a maximum.	The leading coefficient is -4, which makes $a < 0$

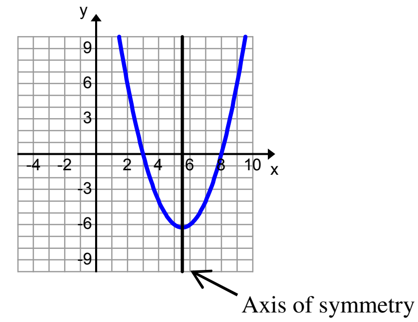
Practice Exercises D

Find the vertex of each equation by completing the square. Determine if the vertex is a maximum or minimum.

1. $f(x) = x^2 + 10x - 20$
2. $f(x) = x^2 - 24x + 1$
3. $f(x) = 5x^2 - 20x - 9$
4. $f(x) = -2x^2 + 16x + 26$
5. $f(x) = x^2 + 8x + 10$
6. $f(x) = -x^2 - 2x - 9$

The **axis of symmetry** is the vertical line that divides a parabola in half. The zeros will always be the same distance from the axis of symmetry.

The vertex always lies on the axis of symmetry.



When completing the square we end up with

$$f(x)=a\left(x+\frac{b}{2a}\right)^2+k$$

$$f(x)=a\left(x-\left(-\frac{b}{2a}\right)\right)^2+k$$

$$f(x)=a\left(x-h\right)^2+k$$

Notice the *x*-coordinate of the vertex is $-\dfrac{b}{2a}$. The *y*-coordinate can be found by evaluating the function at $-\dfrac{b}{2a}$.

Therefore, another method for finding the vertex (*h*, *k*) from a standard form equation is to use $h=\dfrac{-b}{2a}$ and $k=f\left(\dfrac{-b}{2a}\right)$.

Example:

$$f(x)=-3x^2+2x-1$$

$$h=\frac{-12}{2(-3)}=\frac{-12}{-6}=2$$

$$k=f(2)=-3(2)^2+12(2)-1=11$$

The point (2, 11) is the vertex. Since $-3<0$, (2, 11) is the maximum point of the function.

Practice Exercises E

Identify the vertex of each function. Then tell if it is a maximum or minimum point.

1. $f(x)=4x^2+8x-7$
2. $f(x)=x^2-12x+30$
3. $f(x)=2x^2-12x+3$
4. $f(x)=x^2+14x-1$

YOU DECIDE

A model rocket is launched from ground level. The function $h(t) = -16t^2 + 160t$ models the height h (measured in feet) of the rocket after time t (measured in seconds).

Find the zeros and the vertex of the function. Explain what each means in context of the problem.

Practice Exercises F
Solve

1. The height $h(t)$, in feet, of a "weeping willow" firework display, t seconds after having been launched from an 80-ft high rooftop, is given by $h(t) = -16t^2 + 64t + 80$. When will it reach its maximum height? What is its maximum height?

2. The value of some stock can be represented by $V(x) = 2x^2 - 8x + 10$, where x is the number of months after January 2012. What is the lowest value $V(x)$ will reach, and when did that occur?

3. Suppose that a flare is launched upward with an initial velocity of 80 ft/sec from a height of 224 ft. Its height in feet, $h(t)$, after t seconds is given by $h(t) = -16t^2 + 80t + 224$. How long will it take the flare to reach the ground?

4. A company's profit can be modeled by the equation $p(x) = -x^2 + 980x - 3000$ where x is the number of units sold. Find the maximum profit of the company.

5. The Rainbow Bridge Arch at Lake Powell is the world's highest natural arch. The height of an object that has been dropped from the top of the arch can be modeled by the equation $h(t) = -16t^2 + 256$, where t is the time in seconds and h is the height in feet. How long does it take for the object to reach the ground?

6. The amount spent by U.S. companies for online advertising can be approximated by $a(t) = \frac{1}{2}t^2 - 2t + 8$, where $a(t)$ is in billions of dollars and t is the number of years after 2010. In what year after 2010 did U.S. companies spend the least amount of money?

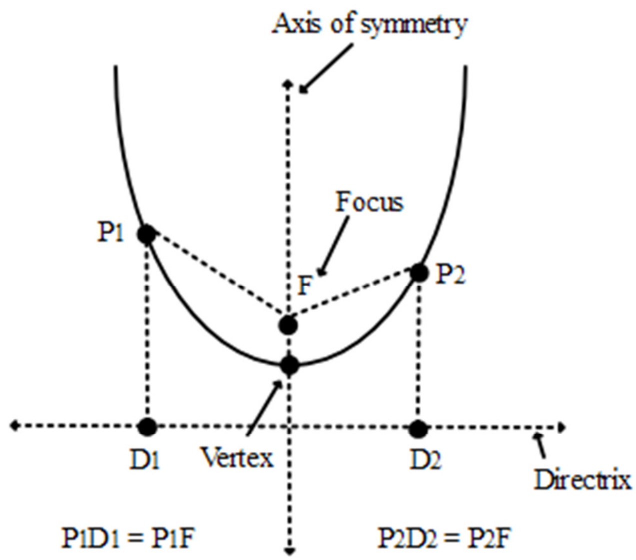
Unit 6 Cluster 3 (G.GPE.2): Parabolas as Conics

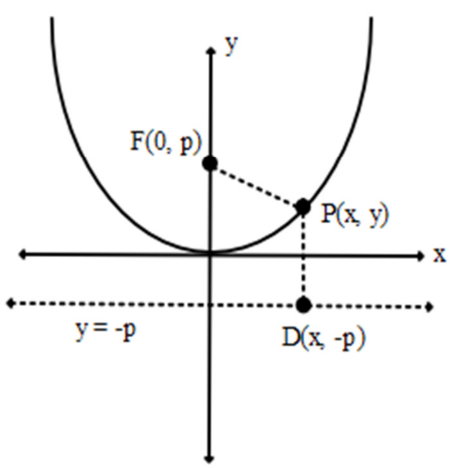
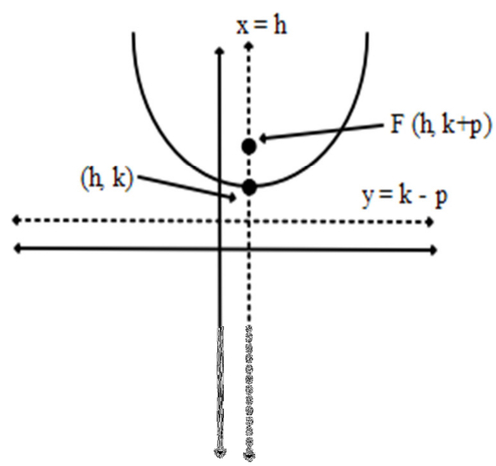
Cluster 3: Translating between descriptions and equations for a conic section

6.3.2 Find the equation of a parabola given the focus and directrix parallel to a coordinate axis.

VOCABULARY

A parabola is the set of all points , $P(x, y)$, in a plane that are an equal distance from both a fixed point, the **focus**, and a fixed line, the **directrix**.



Standard Form for the Equation of a Parabola		
	Vertex at (0, 0)	Vertex at (h, k)
Equation	$y = \frac{1}{4p}x^2$	$y - k = \frac{1}{4p}(x-h)^2$
Direction	Opens upward if $p > 0$ Opens downward if $p < 0$	Opens upward if $p > 0$ Opens downward if $p < 0$
Focus	$(0, p)$	$(h, k + p)$
Directrix	$y = -p$	$y = k - p$
Graph		

Example 1:

Use the Distance Formula to find the equation of a parabola with focus $(0,3)$ and directrix $y = -3$.

$PF = PD$ $\sqrt{(x-x_1)^2+(y-y_1)^2} = \sqrt{(x-x_2)^2+(y-y_2)^2}$	A point $P(x,y)$ on the graph of a parabola is the same distance from the focus $F(0,3)$ and a point on the directrix $D(x,-3)$.
$\sqrt{(x-0)^2+(y-3)^2} = \sqrt{(x-x)^2+(y+3)^2}$	Substitute in known values.
$\sqrt{x^2+(y-3)^2} = \sqrt{(y+3)^2}$	Simplify.
$\left(\sqrt{x^2+(y-3)^2}\right)^2 = \left(\sqrt{(y+3)^2}\right)^2$ $x^2+(y-3)^2 = (y+3)^2$	Square both sides of the equation and use the properties of exponents to simplify.

$x^2 = (y+3)^2 - (y-3)^2$ $x^2 = \left(y^2+6y+9\right) - \left(y^2-6y+9\right)$ $x^2 = 12y$ $y = \frac{1}{12}x^2$	Solve for *y*.

Example:

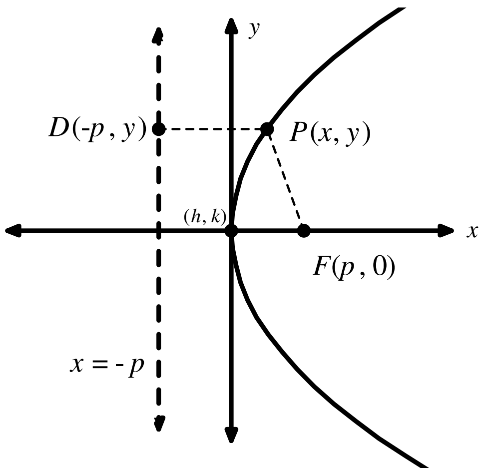
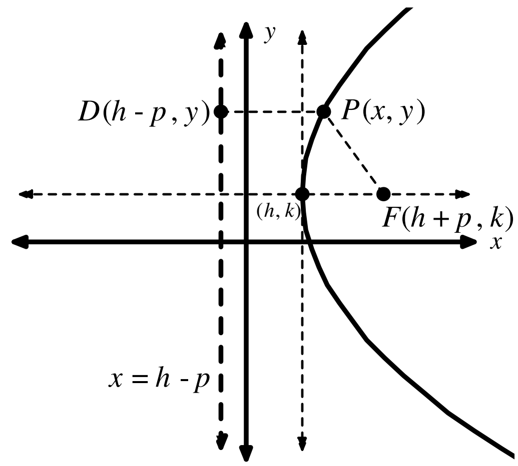
Use the Distance Formula to find the equation of a parabola with focus (-5, 3) and directrix $y = 9$.

$PF = PD$ $\sqrt{(x-x_1)^2+(y-y_1)^2} = \sqrt{(x-x_2)^2+(y-y_2)^2}$	A point $P(x, y)$ on the graph of a parabola is the same distance from the focus $F(-5,3)$ and a point on the directrix $D(x,9)$.
$\sqrt{\left(x-(-5)\right)^2+(y-3)^2} = \sqrt{(x-x)^2+(y-9)^2}$	Substitute in known values.
$\sqrt{(x+5)^2+(y-3)^2} = \sqrt{(y-9)^2}$	Simplify.
$\left(\sqrt{(x+5)^2+(y-3)^2}\right)^2 = \left(\sqrt{(y-9)^2}\right)^2$ $(x+5)^2+(y-3)^2 = (y-9)^2$	Square both sides of the equation and use the properties of exponents to simplify.
$(x+5)^2 = (y-9)^2 - (y-3)^2$ $(x+5)^2 = \left(y^2-18y+81\right) - \left(y^2-6y+9\right)$ $(x+5)^2 = -12y+72$ $(x+5)^2 = -12(y-6)$ $-\frac{1}{12}(x+5)^2 = y-6$	Combine the *x* terms on one side of the equation and the *y* terms on the other side of the equation.

Practice Exercises A

Use the distance formula to find the equation of parabola with the given information.

1. focus $(0,-5)$
   directrix $y=5$
2. focus $(0,7)$
   directrix $y=-7$
3. focus $(0,-3)$
   directrix $y=6$
4. focus $(2,6)$
   directrix $y=-8$
5. focus $(3,4)$
   directrix $y=1$
6. focus $(3,3)$
   directrix $y=7$

Standard Form for the Equation of a Parabola		
	Vertex at (0, 0)	Vertex at (h, k)
Equation	$x = \frac{1}{4p} y^2$	$x - h = \frac{1}{4p}(y-k)^2$
Direction	Opens to the right if $p > 0$ Opens to the left if $p < 0$	Opens to the right if $p > 0$ Opens to the left if $p < 0$
Focus	$(p, 0)$	$(h + p, k)$
Directrix	$x = -p$	$x = h - p$
Graph		

Example:

Use the Distance Formula to find the equation of a parabola with focus $(2, 0)$ and directrix $x = -2$.

$PF = PD$ $\sqrt{(x-x_1)^2 + (y-y_1)^2} = \sqrt{(x-x_2)^2 + (y-y_2)^2}$	A point (x, y) on the graph of a parabola is the same distance from the focus $(2, 0)$ and a point on the directrix $(-2, y)$.
$\sqrt{(x-2)^2 + (y-0)^2} = \sqrt{(x+2)^2 + (y-y)^2}$	Substitute in known values.
$\sqrt{(x-2)^2 + y^2} = \sqrt{(x+2)^2}$	Simplify.
$\left(\sqrt{(x-2)^2 + y^2}\right)^2 = \left(\sqrt{(x+2)^2}\right)^2$ $(x-2)^2 + y^2 = (x+2)^2$	Square both sides of the equation and use the properties of exponents to simplify.

$y^2=(x+2)^2-(x-2)^2$ $y^2=(x^2+4x+4)-(x^2-4x+4)$ $y^2=8x$ $\frac{1}{8}y^2=x$	Solve for *x*.

Example:

Use the Distance Formula to find the equation of a parabola with focus $(4,3)$ and directrix $x=6$.

$PF=PD$ $\sqrt{(x-x_1)^2+(y-y_1)^2}=\sqrt{(x-x_2)^2+(y-y_2)^2}$	A point (x,y) on the graph of a parabola is the same distance from the focus $(4,3)$ and a point on the directrix $(6,y)$.
$\sqrt{(x-4)^2+(y-3)^2}=\sqrt{(x-6)^2+(y-y)^2}$	Substitute in known values.
$\sqrt{(x-4)^2+(y-3)^2}=\sqrt{(x-6)^2}$	Simplify.
$\left(\sqrt{(x-4)^2+(y-3)^2}\right)^2=\left(\sqrt{(x-6)^2}\right)^2$ $(x-4)^2+(y-3)^2=(x-6)^2$	Square both sides of the equation and use the properties of exponents to simplify.
$(y-3)^2=(x-6)^2-(x-4)^2$ $(y-3)^2=x^2-12x+36-(x^2-8x+16)$ $(y-3)^2=-4x+20$ $(y-3)^2=-4(x-5)$ $-\frac{1}{4}(y-3)^2=x-5$	Combine the *x* terms on one side of the equation and the *y* terms on the other side of the equation.

Practice Exercises B

Use the distance formula to find the equation of parabola with the given information.

1. focus $(-4,0)$
directrix $x=4$
2. focus $(5,0)$
directrix $x=-5$
3. focus $(3,0)$
directrix $x=-3$
4. focus $(2,-3)$
directrix $x=5$
5. focus $(-2,-4)$
directrix $x=-6$
6. focus $(-1,1)$
directrix $x=-5$

Practice Exercises C

Determine the vertex, focus, directrix and the direction for each of the following parabolas.

1. $12(y+1)=(x-3)^2$
2. $(x+4)^2=-6(y-1)$
3. $(y-1)^2=-4(x+5)$
4. $6(y-3)=(x+1)^2$
5. $(y+3)^2=12(x-2)$
6. $(y-6)^2=16(x-4)$

You Decide

A parabola has focus (-2,1) and directrix y = -3. Determine whether or not the point (2,1) is part of the parabola. Justify your response.

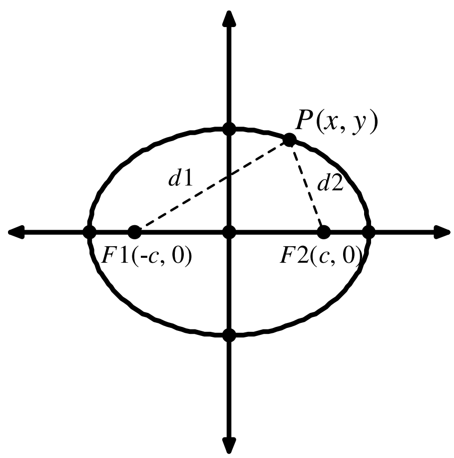
Unit 6 Cluster 3 Honors (G.GPE.3)
Deriving Equations of Ellipses and Hyperbolas

Cluster 3: Translate between the geometric description and the equation for a conic section

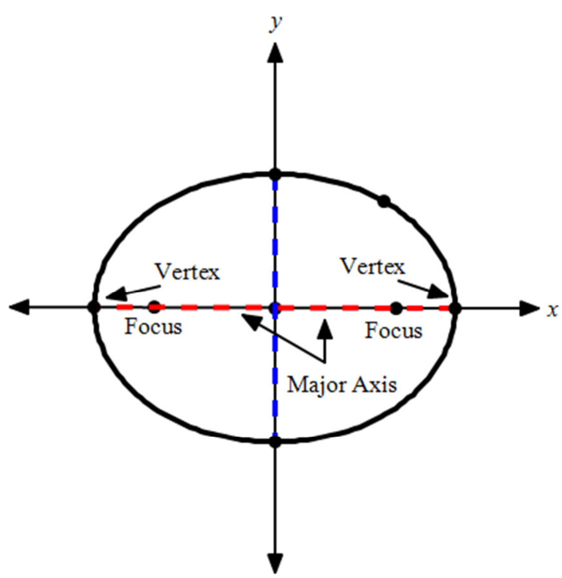
H.5.1 Derive the equations of ellipses and hyperbolas given the foci, using the fact that the sum or difference of distances from the foci is constant.

VOCABULARY

An **ellipse** is the set of all points in a plane the sum of whose distances from two fixed points (called **foci**), F_1 and F_2, is constant. The midpoint of the segments connecting the foci is the **center** of the ellipse.



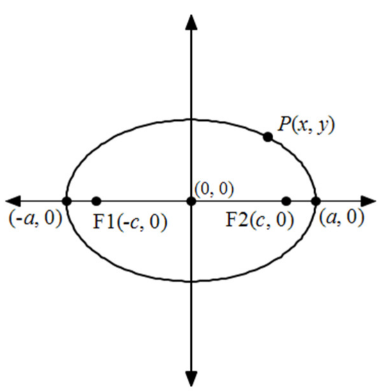
An ellipse can be elongated horizontally or vertically. The line through the foci intersects the ellipse at its **vertices**. The segment whose endpoints are the vertices is called the **major axis**. The **minor axis** is a segment that is perpendicular to the major axis and its endpoints intersect the ellipse.



Deriving the Standard Equation of an Ellipse

$PF_1 + PF_2 = 2a$	The sum of the distance from a point $P(x, y)$ on the ellipse to each foci, $(-c,0)$ and $(c,0)$, is equal to $2a$.
$\sqrt{(x+c)^2+(y-0)^2}+\sqrt{(x-c)^2+(y-0)^2}=2a$	Use the distance formula and substitute in known values.
$\sqrt{(x+c)^2+(y-0)^2}=2a-\sqrt{(x-c)^2+(y-0)^2}$	Isolate one of the radicals.
$\left(\sqrt{(x+c)^2+(y-0)^2}\right)^2=\left(2a-\sqrt{(x-c)^2+(y-0)^2}\right)^2$ $(x+c)^2+(y-0)^2=4a^2-4a\sqrt{(x-c)^2+(y-0)^2}+(x-c)^2+(y-0)^2$ $x^2+2xc+c^2+y^2=4a^2-4a\sqrt{(x-c)^2+(y-0)^2}+x^2-2xc+c^2+y^2$ $4xc=4a^2-4a\sqrt{(x-c)^2+(y-0)^2}$	Square each side then simplify.
$4cx-4a^2=-4a\sqrt{(x-c)^2+(y-0)^2}$ $\frac{4cx-4a^2}{-4}=a\sqrt{(x-c)^2+(y-0)^2}$ $-cx+a^2=a\sqrt{(x-c)^2+(y-0)^2}$ $a^2-cx=a\sqrt{(x-c)^2+(y-0)^2}$	Isolate the radical again.
$(a^2-cx)^2=\left(a\sqrt{(x-c)^2+(y-0)^2}\right)^2$ $a^4-2a^2cx+c^2x^2=a^2\left[(x-c)^2+(y-0)^2\right]$ $a^4-2a^2cx+c^2x^2=a^2\left(x^2-2cx+c^2+y^2\right)$ $a^4-2a^2cx+c^2x^2=a^2x^2-2a^2cx+a^2c^2+a^2y^2$ $a^4+c^2x^2=a^2x^2+a^2c^2+a^2y^2$	Square each side then simplify.
$a^4-a^2c^2=a^2x^2-c^2x^2+a^2y^2$ $a^2\left(a^2-c^2\right)=x^2\left(a^2-c^2\right)+a^2y^2$	Combine all the terms containing x and y on one side.
$a^2b^2=x^2b^2+a^2y^2$	Let $b^2=a^2-c^2$.

$$\frac{a^2b^2}{a^2b^2}=\frac{x^2b^2+a^2y^2}{a^2b^2}$$ $$1=\frac{x^2}{a^2}+\frac{y^2}{b^2}$$	Divide by a^2b^2.

<table>
<tr><th colspan="3">Standard Form for the Equation of an Ellipse Centered at (0, 0)</th></tr>
<tr><td>Equation</td><td>Horizontal Ellipse
$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,\ a^2>b^2$</td><td>Vertical Ellipse
$\frac{x^2}{b^2}+\frac{y^2}{a^2}=1,\ a^2>b^2$</td></tr>
<tr><td>Major Axis</td><td>Along the x-axis
Length: 2a</td><td>Along the y-axis
Length: 2a</td></tr>
<tr><td>Minor Axis</td><td>Along the y-axis
Length: 2b</td><td>Along the x-axis
Length: 2b</td></tr>
<tr><td>Foci</td><td>$(-c,0)$ and $(c,0)$</td><td>$(0,-c)$ and $(0,c)$</td></tr>
<tr><td>Vertices</td><td>$(-a,0)$ and $(a,0)$</td><td>$(0,-a)$ and $(0,a)$</td></tr>
<tr><td>Pythagorean Relation</td><td>$a^2=b^2+c^2$</td><td>$a^2=b^2+c^2$</td></tr>
<tr><td>Basic Graph</td><td>


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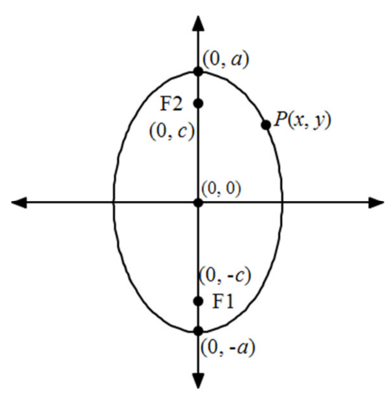


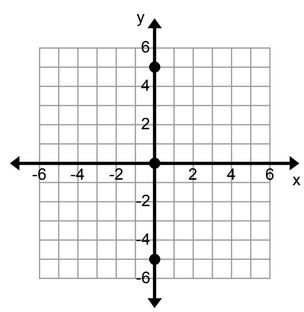
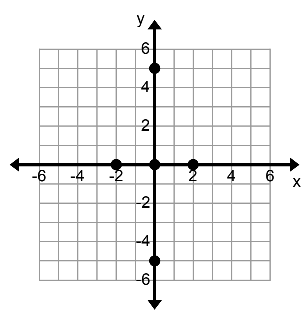
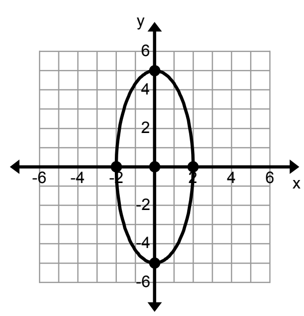
</td></tr>
</table>

Example:

Locate the vertices and foci for the ellipse $25x^2+4y^2=100$. Graph the ellipse.

$25x^2+4y^2=100$	
$$\frac{25x^2}{100}+\frac{4y^2}{100}=\frac{100}{100}$$ $$\frac{x^2}{4}+\frac{y^2}{25}=1$$	The standard equation of an ellipse is equal to 1. Divide each side of the equation by 100 and simplify.

$\frac{x^2}{4}+\frac{y^2}{25}=1$ $a=\sqrt{25}=5$ $b=\sqrt{4}=2$	Identify *a* and *b*. Remember $a^2 > b^2$. Note: *a* and *b* are lengths therefore the positive square root will ALWAYS be used.
$25=4+c^2$ $21=c^2$ $\sqrt{21}=c$	Use *a* and *b* to find *c*. Remember $a^2 = b^2 + c^2$.
Vertices: $(0,-5)$ and $(0,5)$ Foci: $(0,-\sqrt{21})$ and $(0,\sqrt{21})$	The vertices are $(0,-a)$ and $(0,a)$ and the foci are $(0,-c)$ and $(0,c)$ because the ellipse is vertical.

	Begin graphing the ellipse by plotting the center which is at (0, 0). Then plot the vertices which are at $(0,-5)$ and $(0,5)$.
	Use the length of *b* to plot the endpoints of the minor axis. $b=2$ so the endpoints are 2 units to the left and right of the center (0, 0). They are at $(-2,0)$ and $(2,0)$.
	Connect your points with a curve.

Example:

Write an equation in standard form for an ellipse with foci located at $(-2,0)$ and $(2,0)$ and vertices located at $(-6,0)$ and $(6,0)$.

$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$	The ellipse is horizontal because the foci and vertices are along the *x*-axis. Use the standard equation for a horizontal ellipse.
$6^2=b^2+2^2$ $36=b^2+4$ $32=b^2$	$a=6$ $c=2$ Find b^2 using $a^2=b^2+c^2$.
$\frac{x^2}{36}+\frac{y^2}{32}=1$	Substitute in known values.

Practice Exercises A

Locate the vertices and foci of the ellipse, then graph.

1. $\frac{x^2}{16}+\frac{y^2}{7}=1$
2. $\frac{x^2}{21}+\frac{y^2}{25}=1$
3. $\frac{x^2}{27}+\frac{y^2}{36}=1$
4. $3x^2+4y^2=12$
5. $9x^2+4y^2=36$
6. $x^2=1-4y^2$

Write an equation in standard form for the ellipse that satisfies the given conditions.

7. Foci: $(-5,0)$ and $(5,0)$
   Vertices: $(-8,0)$ and $(8,0)$
8. Foci: $(0,-4)$ and $(0,4)$
   Vertices: $(0,-7)$ and $(0,7)$
9. Foci: $(0,-3)$ and $(0,3)$
   Vertices: $(0,-4)$ and $(0,4)$
10. Foci: $(-6,0)$ and $(6,0)$
    Vertices: $(-10,0)$ and $(10,0)$
11. Major axis endpoints: $(0,\pm 6)$
    Minor axis length 8
12. Endpoints of axes are $(\pm 5,0)$ and $(0,\pm 4)$

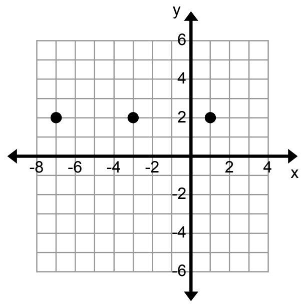
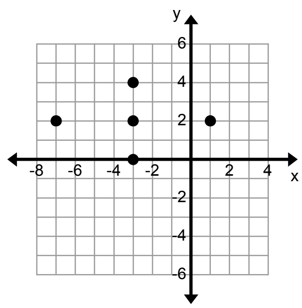
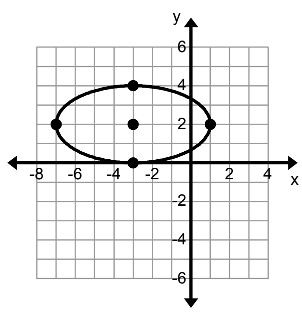
Ellipses Centered at (h, k)

Standard Form for the Equation of an Ellipse Centered at (h, k)		
Equation	$\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1,\ a^2>b^2$	$\frac{(x-h)^2}{b^2}+\frac{(y-k)^2}{a^2}=1,\ a^2>b^2$
Center	(h,k)	(h,k)
Major Axis	Parallel to the x-axis Length: $2a$	Parallel to the y-axis Length: $2a$
Minor Axis	Parallel to the y-axis Length: $2b$	Parallel to the x-axis Length: $2b$
Foci	$(h-c,k)$ and $(h+c,k)$	$(h,k-c)$ and $(h,k+c)$
Vertices	$(h-a,k)$ and $(h+a,k)$	$(h,k-a)$ and $(h,k+a)$
Pythagorean Relation	$a^2=b^2+c^2$	$a^2=b^2+c^2$

Example:

Locate the center, the vertices and the foci of the ellipse $(x+3)^2+4(y-2)^2=16$. Graph the ellipse.

$(x+3)^2+4(y-2)^2=16$	
$\frac{(x+3)^2}{16}+\frac{4(y-2)^2}{16}=\frac{16}{16}$ $\frac{(x+3)^2}{16}+\frac{(y-2)^2}{4}=1$	The standard equation of an ellipse is equal to 1. Divide each side of the equation by 16 and simplify.
$\frac{(x+3)^2}{16}+\frac{(y-2)^2}{4}=1$ $a=\sqrt{16}=4$ $b=\sqrt{4}=2$	Identify a and b. Remember $a^2>b^2$.
$16=4+c^2$ $12=c^2$ $\sqrt{12}=c$ $2\sqrt{3}=c$	Use a and b to find c. Remember $a^2=b^2+c^2$.
Center: $(-3,2)$ Vertices: $(-3-4,2)$ and $(-3+4,2)$ $(-7,2)$ and $(1,2)$ Foci: $(-3-2\sqrt{3},2)$ and $(-3+2\sqrt{3},2)$	$h=-3$ and $k=2$ The ellipse is horizontal, therefore the vertices are $(h-a,k)$ and $(h+a,k)$ and the foci are $(h-c,k)$ and $(h+c,k)$.

	Begin graphing the ellipse by plotting the center of the ellipse $(-3,2)$. Then plot the vertices $(-7,2)$ and $(1,2)$.
	Use the length of b to plot the endpoints of the minor axis. $b=2$ so the endpoints are 2 units above and below the center $(-3,2)$. They are at $(-3,0)$ and $(-3,4)$.
	Connect your points with a curve.

Example:

Write an equation in standard form for an ellipse with foci at $(-2,1)$ and $(-2,5)$ and vertices at $(-2,-1)$ and $(-2,7)$.

$\frac{(x-h)^2}{b^2}+\frac{(y-k)^2}{a^2}=1$	The ellipse is vertical because the foci and vertices are parallel to the y-axis. Use the standard equation for a horizontal ellipse.
$4^2=b^2+2^2$ $16=b^2+4$ $12=b^2$	$2a=7-(-1)$ $\quad 2c=5-1$ $2a=8$ $\quad 2c=4$ $a=4$ $\quad c=2$ Find b^2 using $a^2=b^2+c^2$.
Center: $\left(\frac{-2+-2}{2},\frac{-1+7}{2}\right)=(-2,3)$	The center is the midpoint of the vertices.
$\frac{(x+2)^2}{12}+\frac{(y-3)^2}{16}=1$	Substitute in known values.

Practice Exercises B
Locate the center, vertices and foci of the ellipse, then graph.

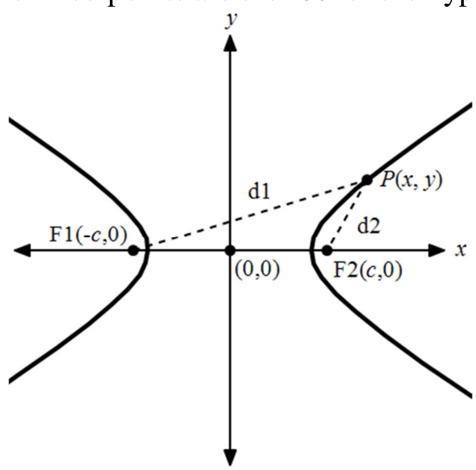
1. $\frac{(x-2)^2}{9}+\frac{(y-1)^2}{4}=1$
2. $\frac{(x-4)^2}{9}+\frac{(y+2)^2}{25}=1$
3. $\frac{(x-3)^2}{9}+\frac{(y+1)^2}{16}=1$
4. $(x-3)^2+9(y+2)^2=18$
5. $9(x-1)^2+4(y+3)^2=36$
6. $2(x+4)^2+4(y+3)^2=24$

Write an equation in standard form for the ellipse that satisfies the given conditions.

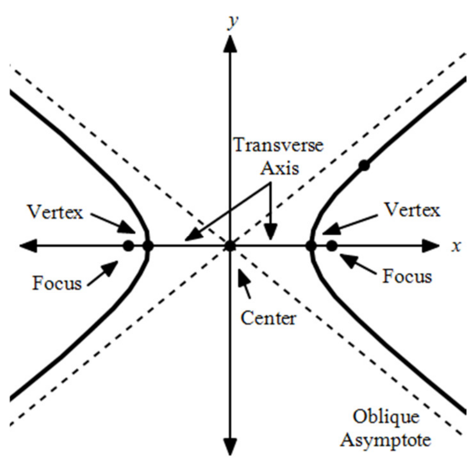
7. Foci: $(1,-4)$ and $(5,-4)$
   Vertices: $(0,-4)$ and $(6,-4)$
8. Foci: $(3,-6)$ and $(3,2)$
   Vertices: $(3,-7)$ and $(3,3)$
9. Foci: $(4,2)$ and $(6,2)$
   Vertices: $(2,2)$ and $(8,2)$
10. Foci: $(1,0)$ and $(1,-4)$
    Vertices: $(1,1)$ and $(1,-5)$
11. Vertices: $(-5,2)$ and $(3,2)$
    Minor axis length is 6.
12. Vertices: $(0,2)$ and $(-6,2)$
    Minor axis length is 2.

VOCABULARY

A **hyperbola** is the set of all points in a plane whose distances from two fixed points in the plane have a constant difference. The fixed points are the **foci** of the hyperbola.



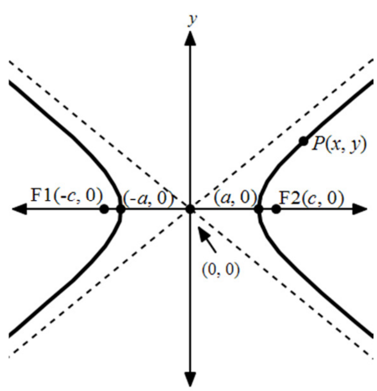
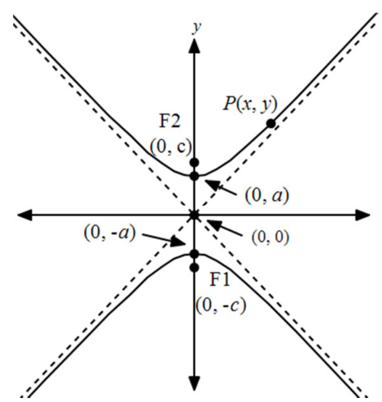
The line through the foci intersects the hyperbola at its **vertices**. The segment connecting the vertices is called the **transverse axis**. The **center** of the hyperbola is the midpoint of the transverse axis. Hyperbolas have two **oblique asymptotes** that intersect at the center.



Deriving the Standard Equation of a Hyperbola

$PF_1 - PF_2 = \pm 2a$	The difference of the distance from a point $P(x, y)$ on the hyperbola to each foci, $(-c, 0)$ and $(c, 0)$, is equal to $\pm 2a$.

$\sqrt{(x-c)^2+(y-0)^2}-\sqrt{(x+c)^2+(y-0)^2}=\pm 2a$ $\sqrt{(x-c)^2+y^2}-\sqrt{(x+c)^2+y^2}=\pm 2a$	Use the distance formula and substitute in known values.
$\sqrt{(x-c)^2+y^2}=\pm 2a+\sqrt{(x+c)^2+y^2}$	Isolate one of the radicals.
$\left(\sqrt{(x-c)^2+y^2}\right)^2=\left(\pm 2a+\sqrt{(x+c)^2+y^2}\right)^2$ $(x-c)^2+y^2=4a^2\pm 4a\sqrt{(x+c)^2+y^2}+(x+c)^2+y^2$ $(x-c)^2+y^2-(x+c)^2-y^2=4a^2\pm 4a\sqrt{(x+c)^2+y^2}$ $x^2-2cx+c^2-\left(x^2+2cx+c^2\right)=4a^2\pm 4a\sqrt{(x+c)^2+y^2}$ $-4cx=4a^2\pm 4a\sqrt{(x+c)^2+y^2}$	Square each side then simplify.
$-4cx-4a^2=\pm 4a\sqrt{(x+c)^2+y^2}$ $\frac{-4cx-4a^2}{4}=\pm a\sqrt{(x+c)^2+y^2}$ $-cx-a^2=\pm a\sqrt{(x+c)^2+y^2}$	Isolate the radical again.
$\left(-cx-a^2\right)^2=\left(\pm a\sqrt{(x+c)^2+y^2}\right)^2$ $c^2x^2+2a^2cx+a^4=a^2\left[(x+c)^2+y^2\right]$ $c^2x^2+2a^2cx+a^4=a^2\left(x^2+2cx+c^2+y^2\right)$ $c^2x^2+2a^2cx+a^4=a^2x^2+2a^2cx+a^2c^2+a^2y^2$ $c^2x^2+a^4=a^2x^2+a^2c^2+a^2y^2$	Square each side then simplify.
$c^2x^2-a^2x^2-a^2y^2=a^2c^2-a^4$ $x^2\left(c^2-a^2\right)-a^2y^2=a^2\left(c^2-a^2\right)$	Combine all the terms containing *x* and *y* on one side.
$x^2b^2-a^2y^2=a^2b^2$	Let $b^2=c^2-a^2$.
$\frac{x^2b^2-a^2y^2}{a^2b^2}=\frac{a^2b^2}{a^2b^2}$ $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$	Divide by a^2b^2.

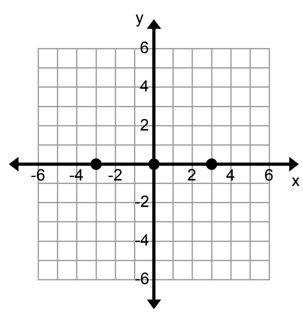
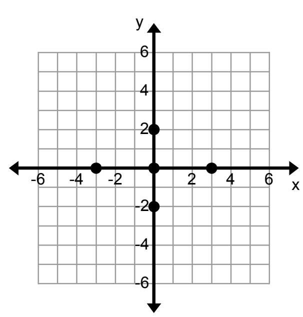
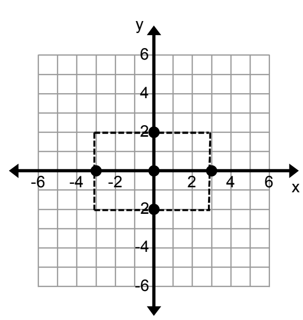
Standard Form for the Equation of a Hyperbola Centered at (0, 0)		
Equation	Opens Left and Right $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$	Opens Up and Down $\frac{y^2}{a^2}-\frac{x^2}{b^2}=1$
Transverse Axis	x-axis Length: 2a	y-axis Length: 2a
Conjugate Axis	y-axis Length:2b	x-axis Length:2b
Foci	$(-c,0)$ and $(c,0)$	$(0,-c)$ and $(0,c)$
Vertices	$(-a,0)$ and $(a,0)$	$(0,-a)$ and $(0,a)$
Pythagorean Relation	$c^2=a^2+b^2$	$c^2=a^2+b^2$
Asymptotes	$y=\pm\frac{b}{a}x$	$y=\pm\frac{a}{b}x$
Basic Graph		

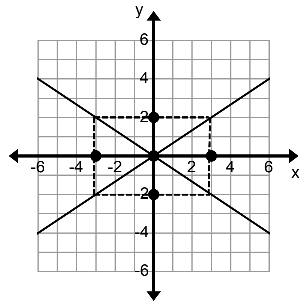
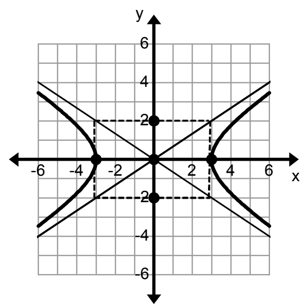
Example:

Find the vertices, foci and asymptotes of the hyperbola $4x^2-9y^2=36$. Then graph the hyperbola.

$4x^2-9y^2=36$	
$\frac{4x^2}{36}-\frac{9y^2}{36}=\frac{36}{36}$ $\frac{x^2}{9}-\frac{y^2}{4}=1$	The standard equation of an ellipse is equal to 1. Divide each side of the equation by 36 and simplify.

$\frac{x^2}{9}-\frac{y^2}{4}=1$ $a=\sqrt{9}=3$ $b=\sqrt{4}=2$	Identify a and b.
$c^2=9+4$ $c^2=13$ $c=\sqrt{13}$	Use a and b to find c. Remember $c^2=a^2+b^2$.
Vertices: $(-3,0)$ and $(3,0)$ Foci: $(-\sqrt{13},0)$ and $(\sqrt{13},0)$ Asymptotes: $y=\frac{2}{3}x$ and $y=-\frac{2}{3}x$	This hyperbola opens left and right so the vertices are $(-a,0)$ and $(a,0)$ and the foci are $(-c,0)$ and $(c,0)$. The asymptotes are $y=\pm\frac{b}{a}x$.

	Begin graphing the hyperbola by plotting the center at (0, 0). Then plot the vertices at $(-3,0)$ and $(3,0)$.
	Use the length of b to plot the endpoints of the conjugate axis. $b=2$ so the endpoints are 2 units above and below the center $(0,0)$. They are at $(0,-2)$ and $(0,2)$.
	Construct a rectangle using the points.

	Draw the asymptotes by drawing a line that connects the diagonal corners of the rectangle and the center.
	Use the asymptotes to help you draw the hyperbola. The hyperbola will open left and right and pass through each vertex.

Example:

Write an equation in standard form for the hyperbola with foci $(0,-3)$ and $(0,3)$ whose conjugate axis has length 4.

$\frac{y^2}{a^2}-\frac{x^2}{b^2}=1$	The foci are along the *y*-axis so the hyperbola's branches open up and down.
$2b=4$ $b=2$	The conjugate axis is length 4. Use it to solve for *b*.
$3^2=a^2+2^2$ $9=a^2+4$ $5=a^2$	Use $b=2$ and $c=3$ to solve for a^2. Remember $c^2=a^2+b^2$.
$\frac{y^2}{5}-\frac{x^2}{4}=1$	Substitute in known values.

Practice Exercises C
Locate the center, vertices, foci and asymptotes of the hyperbola, then graph.

1. $\frac{x^2}{4} - \frac{y^2}{16} = 1$
2. $\frac{y^2}{25} - \frac{x^2}{36} = 1$
3. $\frac{x^2}{1} - \frac{y^2}{9} = 1$
4. $20y^2 - 25x^2 = 100$
5. $4y^2 - 16x^2 = 64$
6. $2x^2 + 4y^2 = 16$

Write an equation in standard form for the hyperbola that satisfies the given conditions.

7. Foci: $(0,-2)$ and $(0,2)$
   Vertices: $(0,-1)$ and $(0,1)$
8. Foci: $(-5,0)$ and $(5,0)$
   Vertices: $(-3,0)$ and $(3,0)$
9. Foci: $(0,-7)$ and $(0,7)$
   Vertices: $(0,-5)$ and $(0,5)$
10. Foci: $(-10,0)$ and $(10,0)$
    Vertices: $(-6,0)$ and $(6,0)$
11. Vertices: $(-4,0)$ and $(4,0)$
    Conjugate axis length is 10.
12. Vertices: $(0,-3)$ and $(0,3)$
    Conjugate axis length is 6.

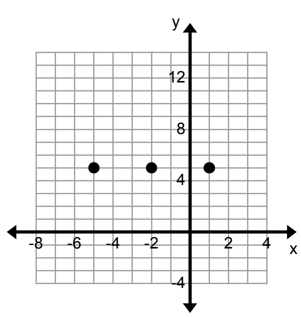
Standard Form for the Equation of a Hyperbola Centered at (h, k)		
Equation	Opens Left and Right $\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$	Opens Up and Down $\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1$
Transverse Axis	Parallel to x-axis Length: $2a$	Parallel to y-axis Length: $2a$
Conjugate Axis	y-axis Length:2b	x-axis Length:2b
Foci	$(h-c,k)$ and $(h+c,k)$	$(h,k-c)$ and $(h,k+c)$
Vertices	$(h-a,k)$ and $(h+a,k)$	$(h,k-a)$ and $(h,k+a)$
Pythagorean Relation	$c^2 = a^2 + b^2$	$c^2 = a^2 + b^2$
Asymptotes	$y-k = \pm\frac{b}{a}(x-h)$	$y-k = \pm\frac{a}{b}(x-h)$

Example:

Find the center, vertices, foci and asymptotes of the hyperbola $\frac{(x+2)^2}{9}-\frac{(y-5)^2}{49}=1$.

Then graph the hyperbola.

$\frac{(x+2)^2}{9}-\frac{(y-5)^2}{49}=1$	
$\frac{(x+2)^2}{9}-\frac{(y-5)^2}{49}=1$ $a=\sqrt{9}=3$ $b=\sqrt{49}=7$	Identify *a* and *b*.
$c^2=9+49$ $c^2=58$ $c=\sqrt{58}$	Use *a* and *b* to find c^2. Remember that $c^2=a^2+b^2$.
Center: $(-2,5)$ Vertices: $(-2-3,5)$ and $(-2+3,5)$ $(-5,5)$ and $(1,5)$ Foci: $\left(-2-\sqrt{58},5\right)$ and $\left(-2+\sqrt{58},5\right)$ Asymptotes: $y-5=\frac{7}{3}(x+2)$ and $y-5=-\frac{7}{3}(x+2)$	The hyperbola's branches open left and right so the vertices are $(h-a,k)$ and $(h+a,k)$. The foci are $(h-c,k)$ and $(h+c,k)$.

<table>
<tr>
<td>
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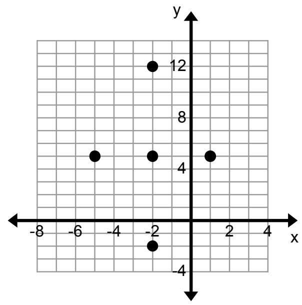
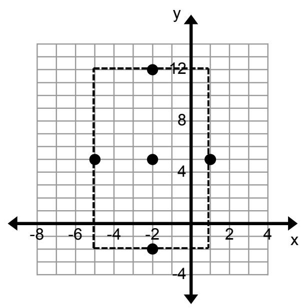
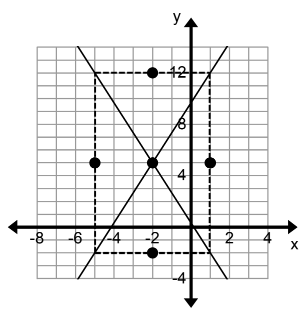
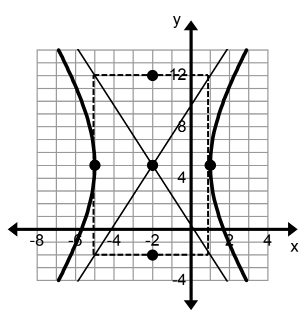
<td>Begin graphing the hyperbola by plotting the center at (-2, 5). Then plot the vertices at $(-5,5)$ and $(1,5)$.</td>
</tr>
</table>

	Use the length of b to plot the endpoints of the conjugate axis. $b = 7$ so the endpoints are 7 units above and below the center $(-2,5)$. They are at $(-2,-2)$ and $(-2,12)$.
	Construct a rectangle using the points.
	Draw the asymptotes by drawing a line that connects the diagonal corners of the rectangle and the center.
	Use the asymptotes to help you draw the hyperbola. The hyperbola will open left and right and pass through each vertex.

Example:

Write an equation in standard form for the hyperbola whose vertices are $(-2,-1)$ and $(8,-1)$ and whose conjugate axis has length 8.

$\frac{(y-k)^2}{a^2}-\frac{(x-h)^2}{b^2}=1$	The foci are parallel to the *x*-axis so the hyperbola's branches open left and right.
Center: $\left(\frac{-2+8}{2},\frac{-1+-1}{2}\right)=(3,-1)$	The midpoint of the vertices is the center of the hyperbola.
$2a=8-(-2)$ $2a=10$ $a=5$	The vertices are at $(-2,-1)$ and $(8,-1)$. Use the distance between them to find *a*.
$2b=8$ $b=4$	The conjugate axis is length 8. Use it to solve for *b*.
$\frac{(x-3)^2}{25}-\frac{(y+1)^2}{16}=1$	Substitute in known values.

Practice Exercises D

Locate the center, vertices, foci and asymptotes of the hyperbola, then graph.

1. $\frac{(y-5)^2}{25}-\frac{(x-6)^2}{16}=1$
2. $\frac{(x+5)^2}{4}-\frac{y^2}{36}=1$
3. $\frac{(x+1)^2}{49}-\frac{(y-3)^2}{16}=1$
4. $4(y+2)^2-(x+6)^2=16$
5. $(y-6)^2-5(x-4)^2=100$
6. $7(x+4)^2+4(y+2)^2=28$

Write an equation in standard form for the hyperbola that satisfies the given conditions.

7. Foci: $(1,9)$ and $(1,1)$
   Vertices: $(1,7)$ and $(1,3)$
8. Foci: $(2,-5)$ and $(-8,-5)$
   Vertices: $(0,-5)$ and $(6,-5)$
9. Foci: $(8,-4)$ and $(-4,-4)$
   Vertices: $(7,-4)$ and $(-3,-4)$
10. Foci: $(-3,5)$ and $(-3,-11)$
    Vertices: $(-3,1)$ and $(-3,-7)$
11. Vertices: $(3,6)$ and $(3,2)$
    Minor axis length is 8.
12. Vertices: $(7,-2)$ and $(-3,-2)$
    Minor axis length is 6.

Unit 2 Cluster 3 (F.BF.1)

Building Functions That Model Relationships Between Two Quantities

Cluster 3: Building functions that model relationships between two quantities

- 2.3.1 Focus on quadratics and exponentials to write a function that describes a relationship between 2 quantities (2^{nd} difference for quadratics)
- 2.3.1 Determine an explicit expression or steps for calculation from context.
- 2.3.1 Combine functions using arithmetic operations.

Vocabulary

A **function** is a relation for which each input has exactly one output. In an ordered pair the first number is considered the input and the second number is considered the output. If any input has more than one output, then the relation is **not** a function.

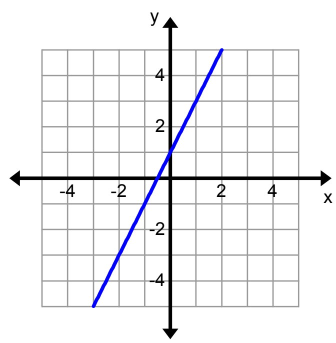
For example the set of ordered pairs {(1,2), (3,5),(8,11)} is a function because each input value has an output value. The set {(1, 2) (1, 3), (6, 7)} does not represent a function because the input 1 has two different outputs 2 and 3.

Linear Function- a function that can be written in the form $y = mx + b$,where *m* and *b* are constants. The graph of a linear function is a line.

A linear function can be expressed in two different ways:

Linear notation: $y = mx + b$

Function notation: $f(x) = mx + b$



$f(x) = 2x + 1$

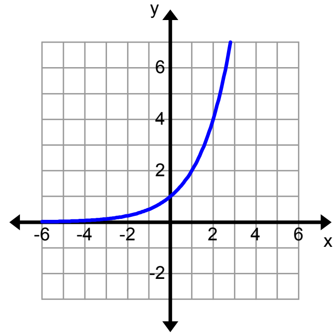
Linear functions can model arithmetic sequences, where the domain is the set of positive integers, because there is a **common difference** between each successive term. The common difference can also be called the **first difference**. Linear functions can model any pattern where the first difference is the same number.

1, 3, 5, 7, …

+2 +2 +2 *1st difference*

Exponential Function- a function of the form $f(x) = ab^x$ where a and b are constants and $a \neq 0$, $b > 0$, and $b \neq 1$.

Exponential functions are most easily recognized by the variable in the exponent. The values of *f(x)* are either increasing (exponential growth) if $a > 0$ and $b > 1$ or decreasing (exponential decay) if $a > 0$ and $0 < b < 1$.



$f(x) = 2^x$

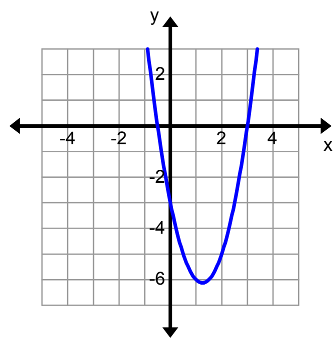
Exponential functions can model geometric sequences, where the domain is the set of positive integers, because each successive term is multiplied by the same number called the **common ratio**. Exponential functions can model any pattern where the next term is obtained by multiplying each successive term by the same number.

1, 3, 9, 27, …

$\times 3$ $\times 3$ $\times 3$ *common ratio*

Quadratic Function- a function that can be written in the form $f(x) = ax^2 + bx + c$ where $a \neq 0$.

Quadratic functions are most easily recognized by the x^2 term. The graph is a parabola. A quadratic function can be formed by multiplying two linear functions. The quadratic function to the right can also be written as $f(x) = (x-3)(2x+1)$.



$f(x) = 2x^2 - 5x - 3$

To determine if a pattern or a sequence can be modeled by a quadratic function, you have to look at the first and **second difference**. The second difference is the difference between the numbers in the first difference. If the first difference is not the same number but the second difference is, then the pattern or sequence can be modeled by a quadratic function.

1, 4, 9, 16, …

+3 +5 +7 *1st difference*

+2 +2 *2nd difference*

Example:

Determine if the pattern 1, 3, 9, 19, … would be modeled by a linear function, an exponential function, or a quadratic function.

Answer:

Check the first difference to see if it is the same number each time. For this pattern, it is not the same, so it will not be modeled by a linear function.

1, 3, 9, 19, ...

+2 +6 +10

Check to see if each term is being multiplied by the same factor. For this pattern, it is not the same, so it will not be modeled by an exponential function.

1, 3, 9, 19, …

×3 ×3 ×2.1

Check the second difference to see if it is the same number each time. For this pattern, it is the same, so the pattern can be modeled by a quadratic function.

1, 3, 9, 19, ...

+2 +6 +10

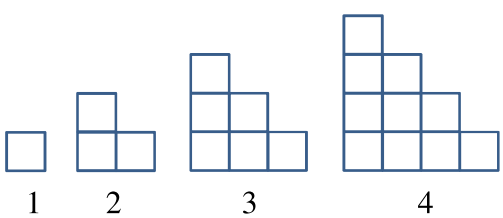
+4 +4

Conclusion: The pattern can be modeled by a quadratic function.

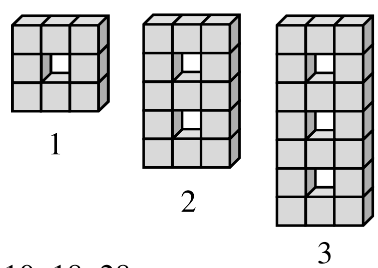
Practice Exercises A

Determine if the pattern would be modeled by a linear function, an exponential function, or a quadratic function.

1.



2.



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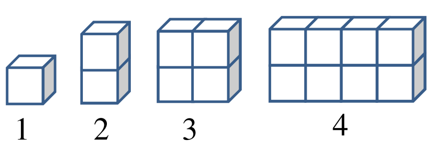


4. 10, 18, 28, …

5. 81, 27, 9, …

6. 8, 16, 24, …

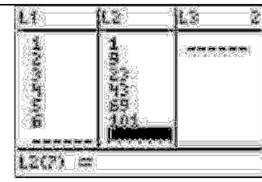
Example:

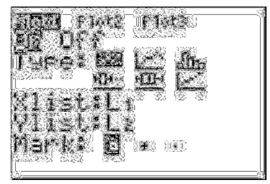
Using a graphing calculator determine the quadratic function modeled by the given data

x	1	2	3	4	5	6
$f(x)$	1	9	23	43	69	101

Input the data into a TI-83 or TI-84 calculator list

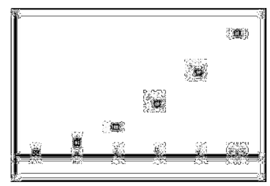
- Enter the information into your lists by pushing STAT followed by Edit.
- If you have values in your lists already, you can clear the information by highlighting the name of the list then pushing CLEAR and ENTER. Do not push DEL or it will delete the entire list.
- Enter the x values into L1 and the $f(x)$ values into L2.
- Push 2nd MODE to get back to the home screen.

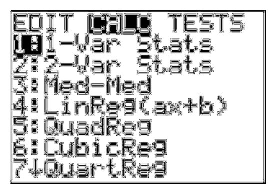




Make a scatter plot

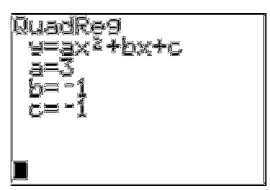
- Push 2nd Y= to bring up the STAT PLOT menu.
- Select Plot1 by pushing ENTER or 1.
- Turn Plot1 on by pushing ENTER when ON is highlighted.
- Make sure that the scatter plot option is highlighted. If it isn't, select it by pushing ENTER when the scatter plot graphic is highlighted.
- The Xlist should say L1 and the Ylist should say L2. If it doesn't, L1 can be entered by pushing 2nd 1 and L2 by 2nd 2.
- To view the graph you can push GRAPH. If you want a nice viewing window, first push ZOOM arrow down to option 9 ZOOMSTAT and either push ENTER or push 9.

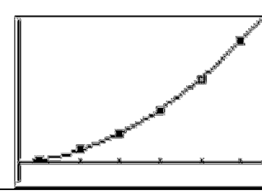




Creating a quadratic regression equation

- You do not have to graph a function to create a regression, but it is recommended that you compare your regression to the data points to determine visually if it is a good model or not.
- From the home screen push STAT, arrow right to CALC and either push 5 for QuadReg or arrow down to 5 and push ENTER. (To do an exponential regression, push 0 for ExpReg or arrow down to 0 and push ENTER.)
- Type 2nd 1, (the comma is located above 7) 2nd 2, VARS arrow right to Y-VARS select FUNCTION and Y1 then push ENTER.
- The quadratic regression is $f(x) = 3x^2 - x - 1$. It has been pasted into Y1 so that you can push GRAPH again and compare your regression to the data.





Practice Exercises B

Find the regression equation. Round to three decimals when necessary.

1. Given the table of values use a graphing calculator to find the quadratic function.

x	0	1	2	3	4	5
$f(x)$	-6	-21	-40	-57	-66	-61

2. Use a graphing calculator to find a quadratic model for the data.

x	1	2	3	4	5	6
$f(x)$	3	1	1	3	7	13

3. From 1972 to 1998 the U.S. Fish and Wildlife Service has kept a list of endangered species in the United States. The table below shows the number of endangered species. Find an appropriate exponential equation to model the data.

Year	1972	1975	1978	1981	1984	1987	1990	1993	1996
Number of species	119.6	157.5	207.3	273	359.4	473.3	623.1	820.5	1080.3

4. The cell phone subscribers of the small town of Herriman are shown below. Find an exponential equation to model the data.

Year	1990	1995	2000	2005	2010
Subscribers	285	802	2,259	6,360	17,904

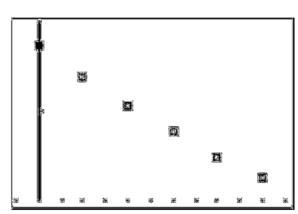
Example:

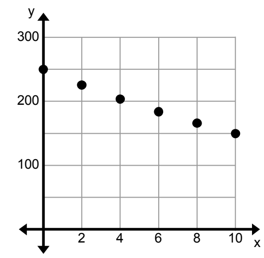
When doctors prescribe medicine, they must consider how much the effectiveness of the drug will decrease as time passes. The table below provides information on how much of the drug remains in a person's system after t hours. Find a model for the data.

t (hours)	0	2	4	6	8	10
Amount (mg)	250	225.6	203.6	183.8	165.9	149.7

Answer:

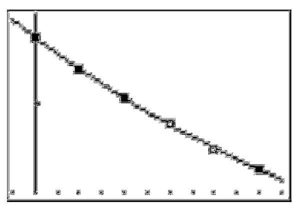
Sometimes it is helpful to look at the graph of the points. For this particular example, it is difficult to determine if this should be modeled by an exponential or a quadratic function

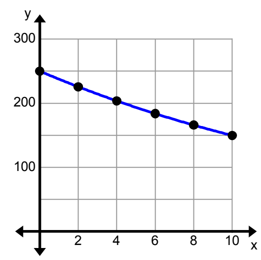




from the graph. Therefore, consider the context of the example. The amount of the drug will continue to decrease unless more is given to the patient. If the patient does not receive more medication, at some point there will only be trace amounts of the drug left in the patient's system. This would suggest a function that continues to decrease until it reaches a leveling off point. An exponential model would be better suited for this situation.

Use the regression capabilities of your graphing calculator to find an exponential model for the data. Follow the instructions for the previous example but make sure that you select option 0: ExpReg. The function that models the data is:
$f(x) = 249.977(0.950)^x$.





Practice Exercises C

Determine if the data is best modeled by an exponential or quadratic function. Then find the appropriate regression equation.

1. The pesticide DDT was widely used in the United States until its ban in 1972. DDT is toxic to a wide range of animals and aquatic life, and is suspected to cause cancer in humans. The half-life of DDT can be 15 or more years. Half-life is the amount of time it takes for half of the amount of a substance to decay. Scientists and environmentalists worry about such substances because these hazardous materials continue to be dangerous for many years after their disposal. Write an equation to model the data below.

Year	1972	1982	1992	2012
Amount of DDT (in grams)	50	9.8	1.9	0.4

2. Use a graphing calculator to find a model for the data.

x	1	2	3	4	5	6
$f(x)$	0	-7	-4	21	80	185

3. The table shows the average movie ticket price in dollars for various years from 1983 to 2003. Find the model for the data.

Years since 1983, t	0	4	8	12	16	20
Movie ticket price, m	4.75	4.07	3.65	4.10	5.08	6.03

4. The table below shows the value of car each year after it was purchased. Find a model for the data.

Years after purchase	0	1	2	3	4	5
Value of car	24,000	20,160	16,934	14,225	11,949	10,037

Combining functions using arithmetic operations

Let *f* and *g* be any two functions. A new function *h* can be created by performing any of the four basic operations on *f* and *g*.

Operation	**Definition**	**Example:** $f(x)=5x^2+2x$, $g(x)=-3x^2$
Addition	$h(x)=f(x)+g(x)$	$h(x)=5x^2+2x+(-3x^2)=2x^2+2x$
Subtraction	$h(x)=f(x)-g(x)$	$h(x)=5x^2+2x-(-3x^2)=8x^2+2x$
Multiplication	$h(x)=f(x)\cdot g(x)$	$h(x)=(5x^2+2x)\cdot(-3x^2)=-15x^4-6x^3$
Division	$h(x)=\dfrac{f(x)}{g(x)}$	$h(x)=\dfrac{5x^2+2x}{-3x^2}=\dfrac{\cancel{x}(5x+2)}{\cancel{x}(-3x)}=\dfrac{5x+2}{-3x}$

The domain of *h* consists of the *x*-values that are in the domains of both *f* and *g*. Additionally, the domain of a quotient does not include *x*-values for which $g(x) = 0$.

Adding and Subtracting Functions

Example:

Let $f(x)=2x+1$ and $g(x)=x^2+3x-4$. Perform the indicated operation and state the domain of the new function.

a. $h(x)=f(x)+g(x)$ b. $h(x)=g(x)-f(x)$ c. $h(x)=2f(x)-g(x)$

Answer:

a.	$h(x)=f(x)+g(x)$	
	$h(x)=(2x+1)+(x^2+3x-4)$	Replace $f(x)$ with $2x+1$ and $g(x)$ with x^2+3x-4.
	$h(x)=2x+1+x^2+3x-4$	Remove the parentheses because we can add in any order.
	$h(x)=x^2+5x-3$	Combine the like terms. $(2x+3x)$, $(1+-4)$
	The domain is $(-\infty,\infty)$.	The domain for both $f(x)$ and $g(x)$ is $(-\infty,\infty)$. $h(x)$ a quadratic function just like $g(x)$ so it has the same domain.

b.	$h(x)=g(x)-f(x)$	
	$h(x)=(x^2+3x-4)-(2x+1)$	Replace $g(x)$ with x^2+3x-4 and $f(x)$ with $2x+1$.
	$h(x)=x^2+3x-4-2x-1$	Distribute the negative throughout the second term and remove the parentheses.
	$h(x)=x^2+x-5$	Combine the like terms.
	The domain is $(-\infty,\infty)$.	The domain for both $f(x)$ and $g(x)$ is $(-\infty,\infty)$. $h(x)$ a quadratic function, just like $g(x)$, so it has the same domain.

c.	$h(x)=2f(x)-g(x)$	
	$h(x)=2(2x+1)-(x^2+3x-4)$	Replace $f(x)$ with $2x+1$ and $g(x)$ with x^2+3x-4.
	$h(x)=4x+2-x^2-3x+4$	Distribute the two through the first term and distribute the negative through the second term.
	$h(x)=-x^2+x+6$	Combine like terms.
	The domain is $(-\infty,\infty)$.	The domain for both $f(x)$ and $g(x)$ is $(-\infty,\infty)$. $h(x)$ a quadratic function just like $g(x)$ so it has the same domain.

Multiplying Functions

Example:

Let $f(x)=2x+1$ and $g(x)=x^2+3x-4$. Perform the indicated operation and state the domain of the new function.

a. $h(x)=f(x)\cdot g(x)$ b. $h(x)=g(x)\cdot g(x)$ C. $h(x)=f(x)\cdot f(x)$

Answer:

a. $h(x)=f(x)\cdot g(x)$	
$h(x)=(2x+1)\cdot(x^2+3x-4)$	Replace $f(x)$ with $2x+1$ and $g(x)$ with x^2+3x-4
$h(x)=2x^3+7x^2-5x-4$	Multiply using your method of choice. (*See Unit 1 Cluster 4 lesson*)
The domain is $(-\infty,\infty)$.	The domain for both $f(x)$ and $g(x)$ is $(-\infty,\infty)$. $h(x)$ a polynomial function just like $g(x)$ so it has the same domain.

b. $h(x)=g(x)\cdot g(x)$	
$h(x)=(x^2+3x-4)\cdot(x^2+3x-4)$	Replace $g(x)$ with x^2+3x-4
$h(x)=x^4+6x^3+x^2-24x+16$	Multiply using your method of choice. (*See Unit 1 Cluster 4 lesson*)
The domain is $(-\infty,\infty)$.	The domain for $g(x)$ is $(-\infty,\infty)$. $h(x)$ a polynomial function just like $g(x)$ so it has the same domain.

c. $h(x)=f(x)\cdot f(x)$	
$h(x)=(2x+1)\cdot(2x+1)$	Replace $f(x)$ with $2x+1$.
$h(x)=4x^2+4x+1$	Multiply using your method of choice. (*See Unit 1 Cluster 4 lesson*)
The domain is $(-\infty,\infty)$.	The domain for $f(x)$ is $(-\infty,\infty)$. $h(x)$ a quadratic function so it has the same domain.

Dividing Functions

VOCABULARY

A **rational function** is a function of the form $r(x)=\dfrac{p(x)}{q(x)}$ where $p(x)$ and $q(x)$ are polynomials and $q(x)\neq 0$. The domain of a rational function includes all real numbers except for those that would make $q(x)=0$.

A rational expression is in **simplified form** if the numerator and the denominator have no common factors other than 1 or -1.

Example:

Let $f(x)=2x+1$ and $g(x)=x^2+3x-4$. Perform the indicated operation and state the domain of the new function.

a. $h(x)=\dfrac{f(x)}{g(x)}$ b. $h(x)=\dfrac{g(x)}{f(x)}$ C. $h(x)=\dfrac{2f(x)}{f(x)}$

Answer:

a. $h(x)=\dfrac{f(x)}{g(x)}$	
$h(x)=\dfrac{2x+1}{x^2+3x-4}$	Replace $f(x)$ with $2x+1$ and $g(x)$ with x^2+3x-4
$h(x)=\dfrac{2x+1}{(x-1)(x+4)}$	Factor the numerator and the denominator to see if the function can be simplified. (*See Unit 2 Cluster 2 (F.IF.8) for help with factoring*)
The domain is $(-\infty,-4)\cup(-4,1)\cup(1,\infty)$. “and”	The new function $h(x)$ is a rational function. The domain cannot include any numbers for which the denominator is zero. The denominator is zero when $x=-4$ and $x=1$.

b. $h(x)=\dfrac{g(x)}{f(x)}$	
$h(x)=\dfrac{x^2+3x-4}{2x+1}$	Replace $f(x)$ with $2x+1$ and $g(x)$ with x^2+3x-4
$h(x)=\dfrac{(x-1)(x+4)}{2x+1}$	Factor the numerator and the denominator to see if the function can be simplified. (*See Unit 2 Cluster 2*

The domain is $\left(-\infty,-\frac{1}{2}\right)\cup\left(-\frac{1}{2},\infty\right)$.	*(F.IF.8) for help with factoring*) The new function $h(x)$ is a rational function. The domain cannot include any numbers for which the denominator is zero. The denominator is zero when $x=-\frac{1}{2}$.

c. $h(x)=\frac{2f(x)}{f(x)}$	
$h(x)=\frac{2(2x+1)}{2x+1}$	Replace $f(x)$ with $2x+1$.
$h(x)=\frac{2\cancel{(2x+1)}}{\cancel{2x+1}}$	Factor the numerator and the denominator to see if the function can be simplified. (*See Unit 2 Cluster 2 (F.IF.8) for help with factoring*)
$h(x)=2$	Divide out the factors and simplify the expression.
The domain is $\left(-\infty,-\frac{1}{2}\right)\cup\left(-\frac{1}{2},\infty\right)$.	Although the simplified form of $h(x)$ is not a rational function, it started out as a rational function and the same restrictions apply on the simplified form. The denominator is zero when $x=-\frac{1}{2}$.

Practice Exercises E

If $f(x)=4x+3$, $g(x)=3x-2$ and $h(x)=12x^2+x-6$, find the following. State the domain of the new function.

1. $f(x)+h(x)$	2. $f(x)+2g(x)$	3. $h(x)-3g(x)$
4. $g(x)+3f(x)-5$	5. $g(x)-h(x)$	6. $h(x)-4g(x)+2$
7. $g(x)\cdot h(x)$	8. $f(x)\cdot h(x)$	9. $h(x)\cdot h(x)$
10. $\frac{f(x)}{h(x)}$	11. $\frac{g(x)}{h(x)}$	12. $\frac{f(x)}{g(x)}$

Evaluating Combined Functions

Example:

Let $f(x) = x + 5$ and $g(x) = 2x^2 + 8x + 5$. Evaluate each expression.

a. $f(2) + g(1)$ b. $f(-3) \cdot g(0)$ c. $\dfrac{g(-1)}{f(-1)}$

Answer:

a. $f(2) + g(1)$	This expression tells you to find the value of f at $x = 2$ and the value of g at $x = 1$ and add the results.
$f(2) = 2 + 5$ $f(2) = 7$	Find the value of f at $x = 2$.
$g(1) = 2(1)^2 + 8(1) + 5$ $g(1) = 2 \cdot 1 + 8 + 5$ $g(1) = 2 + 8 + 5$ $g(1) = 15$	Find the value of g at $x = 1$.
$7 + 15 = 22$ $f(2) + g(1) = 22$	Add the results.

b. $f(-3) \cdot g(0)$	This expression tells you to find the value of f at $x = -3$ and the value of g at $x = 0$ and multiply the results.
$f(-3) = -3 + 5$ $f(-3) = 2$	Find the value of f at $x = -3$.
$g(0) = 2(0)^2 + 8(0) + 5$ $g(0) = 2 \cdot 0 + 0 + 5$ $g(0) = 0 + 0 + 5$ $g(0) = 5$	Find the value of g at $x = 0$.
$2 \cdot 5 = 10$ $f(-3) \cdot g(0) = 10$	Multiply the results.

c. $\dfrac{g(-1)}{f(-1)}$	This expression tells you to find the value of g at $x=-1$ and the value of f at $x=-1$ and divide the results.
$g(-1)=2(-1)^2+8(-1)+5$ $g(-1)=2\cdot 1-8+5$ $g(-1)=2-8+5$ $g(-1)=-1$	Find the value of g at $x=-1$.
$f(-1)=-1+5$ $f(-1)=4$	Find the value of f at $x=-1$.
$\dfrac{-1}{4}$ $\dfrac{g(-1)}{f(-1)}=-\dfrac{1}{4}$ or -0.25	Divide the results.

Practice Exercises F

If $f(x)=3x+1$ and $g(x)=3x^2-5x-2$, find the value of each expression.

1. $5f(2)$
4. $2f(1)-g(5)$
7. $f(-2)-g(4)$
2. $g(-1)+f(3)$
5. $f(2)\cdot g(4)$
8. $f(-1)\cdot 3g(1)$
3. $\dfrac{f(3)}{f(-3)}$
6. $\dfrac{g(1)}{f(4)}$
9. $\dfrac{g(0)}{2f(0)}$

Practice Exercises G

1. A company estimates that its cost and revenue can be modeled by the functions $C(x) = 0.75x^2 + 100x + 20{,}000$ and $R(x) = 150x + 100$ where x is the number of units produced. The company's profit, P, is modeled by $R(x) - C(x)$. Find the profit equation and determine the profit when 1,000,000 units are produced.

2. Consider the demand equation $p(x) = -\frac{1}{15}x + 30;\ 0 \le x \le 450$ where p represents the price and x the number of units sold. Write an equation for the revenue, R, if the revenue is the price times the number of units sold. What price should the company charge to have maximum revenue?

3. The average Cost $\overline{C}$ of manufacturing x computers per day is obtained by dividing the cost function by the number of computers produced that day, x. If the cost function is $C(x) = 0.5x^3 - 34x^2 + 1213x$, find an equation for the average cost of manufacturing. What is the average cost of producing 100 computers per day?

4. The service committee wants to organize a fund-raising dinner. The cost of renting a facility is \$300 plus \$5 per chair or $C(x) = 5x + 300$, where x represents the number of people attending the fund-raiser. The committee wants to charge attendees \$30 each or $R(x) = 30x$. How many people need to attend the fund-raiser for the event to raise \$1,000?

Unit 2 Cluster 4 (F.BF.3 and F.BF.4): Transformations and Inverses

Cluster 4: Building New Functions from Existing Functions

- 2.4.1 Transformations, odd and even graphically and algebraically
- 2.4.2 Find inverse functions (simple) focus on linear and basic restrictions for quadratics, introduce one-to-one and horizontal line test

VOCABULARY

There are several types of functions (linear, exponential, quadratic, absolute value, etc.). Each of these could be considered a family. Each family has their own unique characteristics that are shared among the members. The **parent function** is the basic function that is used to create more complicated functions.

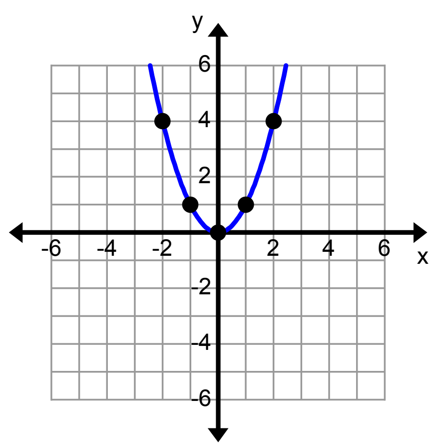
The graph of a quadratic function is in the shape of a **parabola**. This is generally described as being "u" shaped.

The maximum or minimum point of a quadratic function is the **vertex**. When a quadratic function is written in **vertex form,** $f(x) = a(x-h)^2 + k$, then the vertex, (h, k), is highlighted.

The **axis of symmetry** is the vertical line that divides the graph in half, with each half being a reflection of the other. The equation for the axis of symmetry is $x = h$.

Quadratic parent function $f(x) = x^2$

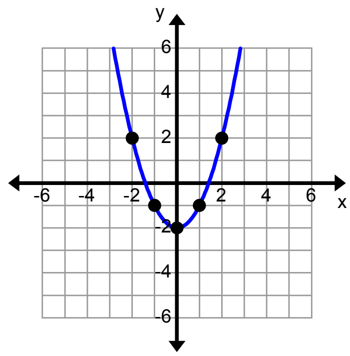
x	$f(x) = x^2$
-3	$(-3)^2 = 9$
-2	$(-2)^2 = 4$
-1	$(-1)^2 = 1$
0	$(0)^2 = 0$
1	$(1)^2 = 1$
2	$(2)^2 = 4$
3	$(3)^2 = 9$



The axis of symmetry is the line $x = 0$. The vertex is the point (0, 0). The domain is the set of all real numbers $(-\infty, \infty)$. The range is the set of positive real numbers including zero $[0, \infty)$.

Vertical Shift: $f(x) = x^2 - 2$

x	$f(x) = x^2 - 2$
-3	$(-3)^2 - 2 = 7$
-2	$(-2)^2 - 2 = 2$
-1	$(-1)^2 - 2 = -1$
0	$(0)^2 - 2 = -2$
1	$(1)^2 - 2 = -1$
2	$(2)^2 - 2 = 2$
3	$(3)^2 - 2 = 7$



Axis of symmetry: $x = 0$

Vertex: (0, -2)

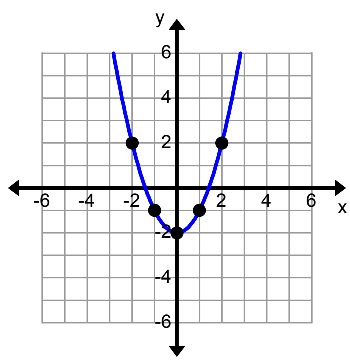
Domain: $(-\infty, \infty)$

Range: $[-2, \infty)$

Effect on the graph: The parabola has been shifted down 2 units.

Vertical Shift: $f(x) = x^2 + 1$

x	$f(x) = x^2 + 1$
-3	$(-3)^2 + 1 = 10$
-2	$(-2)^2 + 1 = 5$
-1	$(-1)^2 + 1 = 2$
0	$(0)^2 + 1 = 1$
1	$(1)^2 + 1 = 2$
2	$(2)^2 + 1 = 5$
3	$(3)^2 + 1 = 10$



Axis of symmetry: $x = 0$

Vertex: (0, 1)

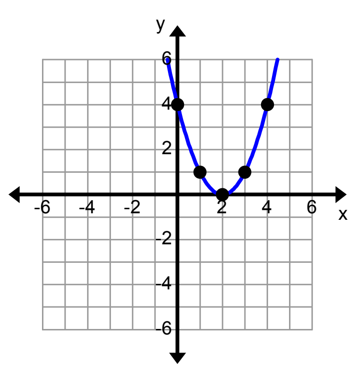
Domain: $(-\infty, \infty)$

Range: $[1, \infty)$

Effect on the graph: The parabola has been shifted up 1 unit.

Horizontal Shift: $f(x) = (x-2)^2$

x	$f(x) = (x-2)^2$
-1	$(-1-2)^2 = 9$
0	$(0-2)^2 = 4$
1	$(1-2)^2 = 1$
2	$(2-2)^2 = 0$
3	$(3-2)^2 = 1$
4	$(4-2)^2 = 4$
5	$(5-2)^2 = 9$



Axis of symmetry: $x = 2$

Vertex: (2, 0)

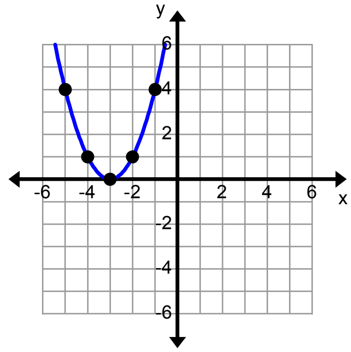
Domain: $(-\infty, \infty)$

Range: $[0, \infty)$

Effect on the graph: the parabola has been shifted 2 units to the right.

Horizontal Shift: $f(x) = (x+3)^2$

x	$f(x) = (x+3)^2$
-6	$(-6+3)^2 = 9$
-5	$(-5+3)^2 = 4$
-4	$(-4+3)^2 = 1$
-3	$(-3+3)^2 = 0$
-2	$(-2+3)^2 = 1$
-1	$(-1+3)^2 = 4$
0	$(0+3)^2 = 9$



Axis of symmetry: $x = -3$

Vertex: (-3, 0)

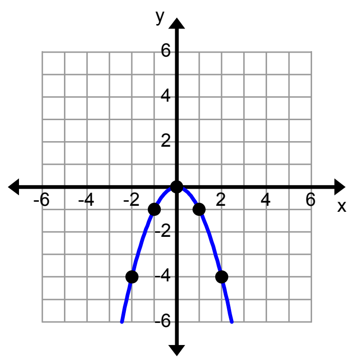
Domain: $(-\infty, \infty)$

Range: $[0, \infty)$

Effect on the graph: the parabola has been shifted three units to the left.

Reflection: $f(x) = -x^2$

x	$f(x) = -x^2$
-3	$-(-3)^2 = -9$
-2	$-(-2)^2 = -4$
-1	$-(-1)^2 = -1$
0	$-(0)^2 = 0$
1	$-(1)^2 = -1$
2	$-(2)^2 = -4$
3	$-(3)^2 = -9$



Axis of symmetry: $x = 0$

Vertex: (0, 0)

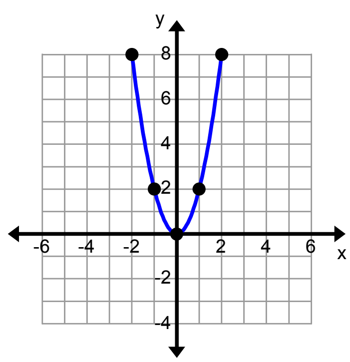
Domain: $(-\infty, \infty)$

Range: $(-\infty, 0]$

Effect on the graph: the parabola has been reflected over the *x*-axis.

Vertical Stretch: $f(x) = 2x^2$

x	$f(x) = 2x^2$
-3	$2(-3)^2 = 18$
-2	$2(-2)^2 = 8$
-1	$2(-1)^2 = 2$
0	$2(0)^2 = 0$
1	$2(1)^2 = 2$
2	$2(2)^2 = 4$
3	$2(3)^2 = 18$



Axis of symmetry: $x = 0$

Vertex: (0, 0)

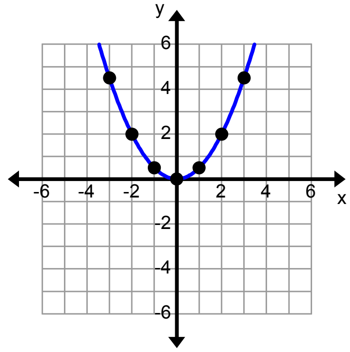
Domain: $(-\infty, \infty)$

Range: $[0, \infty)$

Effect on the graph: the *y*-coordinates of the parabola have been multiplied by 2.

Vertical Shrink: $f(x)=\frac{1}{2}x^2$

x	$f(x)=\frac{1}{2}x^2$
-3	$\frac{1}{2}(-3)^2=\frac{9}{2}$
-2	$\frac{1}{2}(-2)^2=2$
-1	$\frac{1}{2}(-1)^2=\frac{1}{2}$
0	$\frac{1}{2}(0)^2=0$
1	$\frac{1}{2}(1)^2=\frac{1}{2}$
2	$\frac{1}{2}(2)^2=4$
3	$\frac{1}{2}(-3)^2=\frac{9}{2}$



Axis of symmetry: $x=0$

Vertex: (0, 0)

Domain: $(-\infty,\infty)$

Range: $[0,\infty)$

Effect on the graph: the y-coordinates of the parabola have been divided 2.

Horizontal Shift

$$f(x)=a(x-h)^2+k$$

Vertical Stretch or Reflection

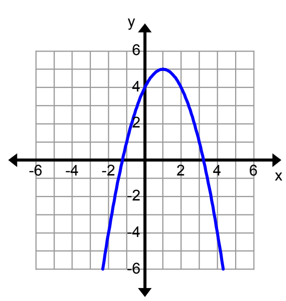
Vertical Shift

Example:

Describe the transformations performed on $f(x)=x^2$ to make it $f(x)=-(x-1)^2+5$. Then graph the function and identify the axis of symmetry, the vertex, the domain and the range.

Transformations:

- reflected over the x-axis
- shifted one unit to the right
- shifted up three units



Axis of symmetry: $x=1$

Vertex: (1, 5)

Domain: $(-\infty,\infty)$

Range: $(-\infty,5]$

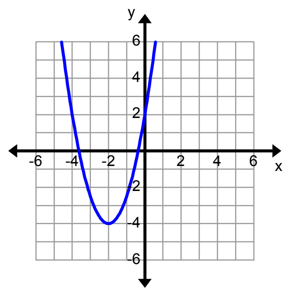
Example:

Describe the transformations performed on $f(x)=x^2$ to make it $f(x)=\frac{3}{2}(x+2)^2-4$.

Then graph the function and identify the axis of symmetry, the vertex, the domain and the range.

Transformations:

- y-coordinates multiplied by 3/2
- shifted two units to the left
- shifted down four units



Axis of symmetry: $x=-2$

Vertex: (-2, -4)

Domain: $(-\infty,\infty)$

Range: $[-4,\infty)$

Practice Exercises A

Describe the transformations performed on $f(x)=x^2$ to make it the following:

1. $f(x)=-x^2+6$
2. $f(x)=(x+5)^2-7$
3. $f(x)=3(x-4)^2$

Graph each function and identify the axis of symmetry, the vertex, the domain and the range.

4. $f(x)=(x-2)^2-6$
5. $f(x)=-2(x-1)^2-3$
6. $f(x)=-\frac{1}{2}x^2+2$
7. $f(x)=5(x+6)^2-4$
8. $f(x)=3x^2-4$
9. $f(x)=\frac{3}{2}(x-3)^2$

VOCABULARY

The **absolute value function** is actually a piecewise-defined function consisting of two linear equations.

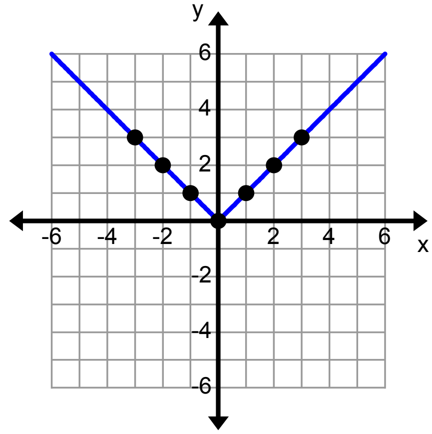
$$f(x)=|x|=\begin{cases} x, & \text{if } x\geq 0 \\ -x, & \text{if } x<0 \end{cases}$$

Absolute value is often defined as the distance from zero. Therefore, the output is positive.

The point where the two linear equations meet is called the **vertex**. It is also the minimum or maximum of the function. The vertex, (h,k), can easily be identified when the absolute value function is represented in the form $f(x)=a|x-h|+k$.

Absolute Value parent function $f(x)=|x|$

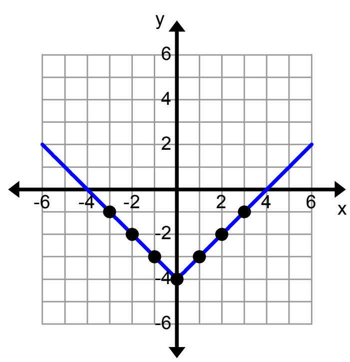
x	$f(x)=\lvert x\rvert$
-3	$\lvert -3\rvert =3$
-2	$\lvert -2\rvert =2$
-1	$\lvert -1\rvert =1$
0	$\lvert 0\rvert =0$
1	$\lvert 1\rvert =1$
2	$\lvert 2\rvert =2$
3	$\lvert 3\rvert =3$



The vertex is the point (0, 0). The domain is the set of all real numbers $(-\infty,\infty)$. The range is the set of positive real numbers including zero $[0,\infty)$.

Vertical Shift: $f(x)=|x|-4$

x	$f(x)=\lvert x\rvert -4$
-3	$\lvert -3\rvert -4=-1$
-2	$\lvert -2\rvert -4=-2$
-1	$\lvert -1\rvert -4=-3$
0	$\lvert 0\rvert -4=-4$
1	$\lvert 1\rvert -4=-3$
2	$\lvert 2\rvert -4=-2$
3	$\lvert 3\rvert -4=-1$



Vertex: (0, -4)

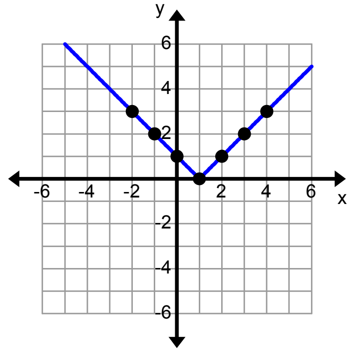
Domain: $(-\infty,\infty)$

Range: $[-4,\infty)$

Effect on the graph: The function has been shifted down 4 units.

Horizontal Shift: $f(x)=|x-1|$

x	$f(x)=\lvert x-1\rvert$
-2	$\lvert -2-1\rvert=3$
-1	$\lvert -1-1\rvert=2$
0	$\lvert 0-1\rvert=1$
1	$\lvert 1-1\rvert=0$
2	$\lvert 2-1\rvert=1$
3	$\lvert 3-1\rvert=2$
4	$\lvert 4-1\rvert=3$



Vertex: (1, 0)

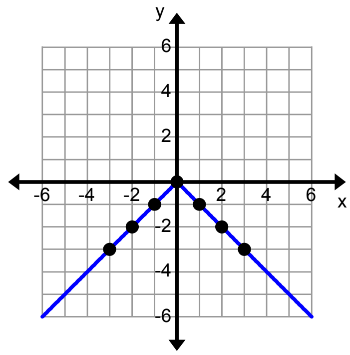
Domain: $(-\infty,\infty)$

Range: $[0,\infty)$

Effect on the graph: The function has been shifted right 1 unit.

Reflection: $f(x)=-|x|$

x	$f(x)=-\lvert x\rvert$
-3	$-\lvert -3\rvert=-3$
-2	$-\lvert -2\rvert=-2$
-1	$-\lvert -1\rvert=-1$
0	$-\lvert 0\rvert=0$
1	$-\lvert 1\rvert=-1$
2	$-\lvert 2\rvert=-2$
3	$-\lvert 3\rvert=-3$



Vertex: (0, 0)

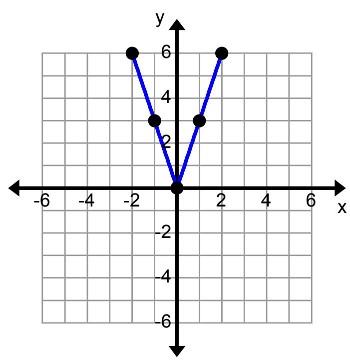
Domain: $(-\infty,\infty)$

Range: $(-\infty,0]$

Effect on the graph: The function has been reflected over the *x*-axis.

Vertical Stretch: $f(x)=3|x|$

x	$f(x)=3\lvert x\rvert$
-3	$3\lvert -3\rvert=9$
-2	$3\lvert -2\rvert=6$
-1	$3\lvert -1\rvert=3$
0	$3\lvert 0\rvert=0$
1	$3\lvert 1\rvert=3$
2	$3\lvert 2\rvert=6$
3	$3\lvert 3\rvert=9$



Vertex: (0, 0)

Domain: $(-\infty,\infty)$

Range: $[0,\infty)$

Effect on the graph: The *y*-coordinates of the function have been multiplied by 3.

$$f(x) = \underset{\substack{\uparrow \\ \text{Vertical Stretch} \\ \text{or Reflection}}}{a}\left|x - \overset{\substack{\text{Horizontal Shift} \\ \downarrow}}{h}\right| + \underset{\substack{\uparrow \\ \text{Vertical Shift}}}{k}$$

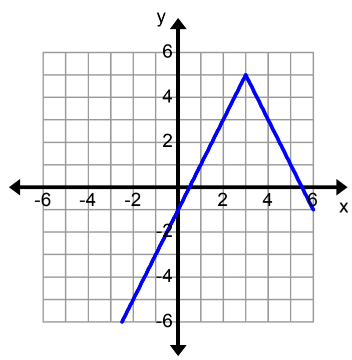
Example:

Describe the transformations performed on $f(x) = |x|$ to make it $f(x) = -2|x-3|+5$.

Then graph the function and identify the axis of symmetry, the vertex, the domain and the range.

Transformations:

- reflected over the *x*-axis
- *y*-coordinates multiplied by 2
- shifted three units to the right
- shifted up five units



Vertex: (3, 5)

Domain: $(-\infty, \infty)$

Range: $(-\infty, 5]$

Practice Exercises B

Describe the transformations performed on $f(x) = |x|$ to make it the following:

1. $f(x) = -2|x|+5$
2. $f(x) = |x-3|+4$
3. $f(x) = 3|x+2|-5$

Graph each function and identify the vertex, the domain and the range.

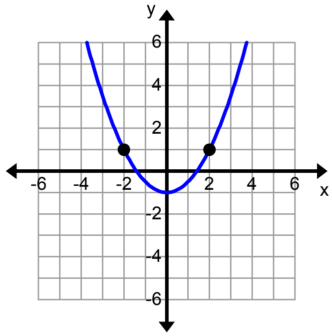
4. $f(x) = -|x|+6$
5. $f(x) = |x+2|-4$
6. $f(x) = -\frac{1}{2}|x-1|$
7. $f(x) = 2|x|-5$
8. $f(x) = |x-3|+1$
9. $f(x) = \frac{3}{2}|x+4|$

Vocabulary

The words **even** and **odd** describe the symmetry that exists for the graph of a function.

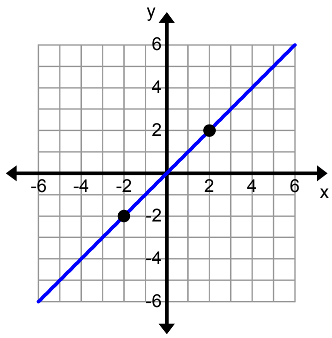
A function is considered to be **even** if, for every number x in its domain, the number $-x$ is also in the domain and $f(-x) = f(x)$. Even functions have y-axis symmetry.

A function is considered to be **odd** if, for every number x in its domain, the number $-x$ is also in the domain and $f(-x) = -f(x)$. Odd functions have origin symmetry.



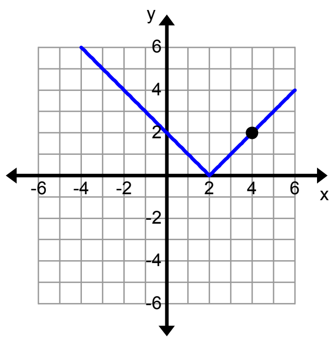
Even Function

The function graphed at the left is even because (2, 1) is a point on the graph and (-2, 1) is also a point on the graph. Notice that -2 is the opposite of 2, but both inputs give the same output. Therefore, $f(-x) = f(x)$, i.e. opposite inputs generate the same output.



Odd Function

The function graphed at the left is odd because (2, 2) is a point on the graph and (-2, -2) is also a point on the graph. Notice that the input -2 is the opposite of 2, and gives the opposite output from 2. Therefore, $f(-x) = -f(x)$, i.e. opposite inputs generate outputs that are opposites of each other.



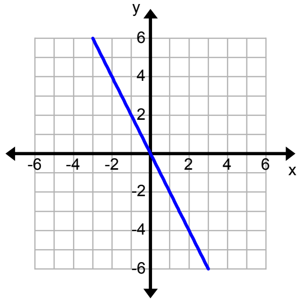
Neither Even nor Odd

The function graphed at the left is neither even nor odd. It is not even because the point (4, 2) is on the graph, but (-4, 2) is not. Similarly, it is not odd because the point (-4, -2) is not a point on the graph.

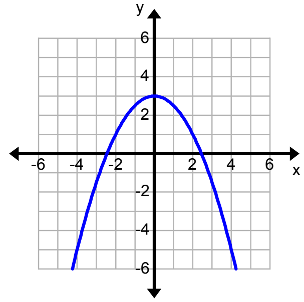
Practice Exercises C

Determine if the following graphs represent functions that are even, odd or neither.

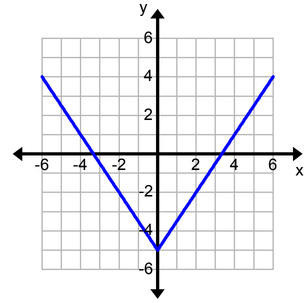
1.



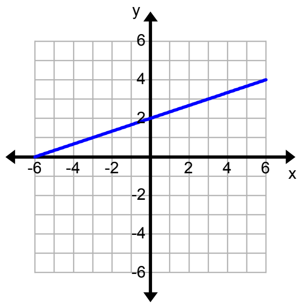
2.



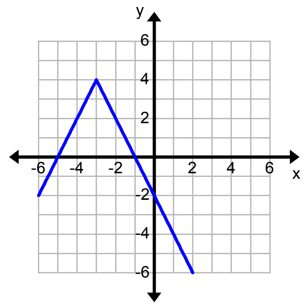
3.



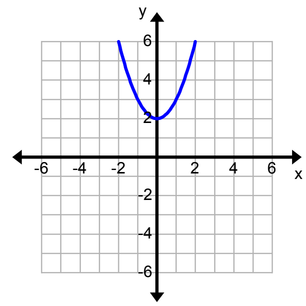
4.



5.



6.



Determining Even and Odd Algebraically

It is possible to graph functions and visually determine whether the function is even or odd, but there is also an algebraic test that can be applied. It was previously stated that if a function is **even**, then evaluating the function at x and $-x$ should produce the same output or $f(-x) = f(x)$. If a function is **odd**, then evaluating the function at x and $-x$ should produce outputs that are opposite or $f(-x) = -f(x)$.

Example:

Is the function $f(x) = -3x$ an even function, an odd function, or neither?

Original function.	$f(x) = -3x$
Substitute $-x$ in for each x in the function.	$f(-x) = -3(-x)$
Simplify.	$f(-x) = 3x$
Compare the output to the original function.	$3x \neq -3x$
If they are the same, then the function is even. If they are opposite, then the function is odd. If they are anything else, then they are neither.	$f(-x) \neq f(x)$ $f(-x) = -f(x)$

Conclusion: $f(x) = -3x$ is an odd function.

Example:

Is the function $g(x) = x^2 - 4x + 4$ an even function, an odd function or neither?

Original function.	$g(x) = x^2 - 4x + 4$
Substitute $-x$ in for each x in the function.	$g(-x) = (-x)^2 - 4(-x) + 4$
Simplify.	$g(-x) = x^2 + 4x + 4$
Compare the output to the original function.	$x^2 + 4x + 4 \neq x^2 - 4x + 4$
If they are the same, then the function is even. If they are opposite, then the function is odd. If they are anything else, then they are neither.	$g(-x) \neq g(x)$ $g(-x) \neq -g(x)$

Conclusion: $g(x) = x^2 - 4x + 4$ is neither an even nor an odd function.

Example:

Is the function $h(x) = -2|x| + 4$ an even function, an odd function, or neither?

Original function.	$h(x) = -2|x| + 4$
Substitute $-x$ in for each x in the function.	$h(x) = -2|(-x)| + 4$
Simplify.	$h(x) = -2|x| + 4$
Compare the output to the original function.	$-2|x| + 4 = -2|x| + 4$
If they are the same, then the function is even. If they are opposite, then the function is odd. If they are anything else, then they are neither.	$h(-x) = h(x)$ $h(-x) \neq -h(x)$

Conclusion: $h(x) = -2|x| + 4$ is an even function.

Practice Exercises D

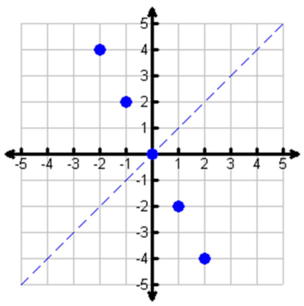
Determine algebraically if the function is even, odd, or neither.

1. $f(x) = |x+3| - 1$
2. $f(x) = \frac{1}{3}x$
3. $f(x) = -3x^2$
4. $f(x) = 4x - 5$
5. $f(x) = -|x| + 4$
6. $f(x) = \frac{5}{4}(x-1)^2$

VOCABULARY

The graph of an **inverse relation** is the *reflection* of the graph of the original relation. The line of reflection is $y = x$.

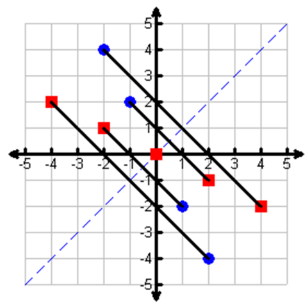
The original relation is the set of ordered pairs: {(-2, 1), (-1, 2), (0, 0), (1, -2), (2, -4)}. The inverse relation is the set of ordered pairs: {(1, -2), (2, -1), (0, 0), (-2, 1), (-4, 2)}. Notice that for the inverse relation the domain (x) and the range (y) reverse positions.



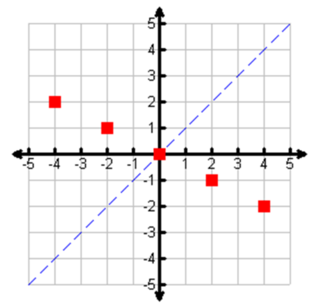
Original Relation

Domain: {-2, -1, 0, 1, 2}

Range: {-4, -2, 0, 1, 2}



The points are reflected over the line $y = x$. Notice that each point is the same distance away from the line, but on the opposite side of the line.



Inverse Relation

Domain: {-4, -2, 0, 1, 2}

Range: {-2, -1, 0, 1, 2}

Practice Exercises E

Find the inverse relation.

1. {(1, -1), (2, -2), (3, -3), (4, -4), (5, -5)}

2. {(-4, 2), (-2, 1), (0, 0), (2, 1), (4, 2)}

3. {(-10, 6), (3, -9), (-1, 4), (-7, 1), (6, 8)}

4. {(7, 6), (2, 9), (-3, -2), (-7, 1), (8, 10)}

VOCABULARY

If no vertical line intersects the graph of a function f more than once, then f is a function. This is called the **vertical line test**.

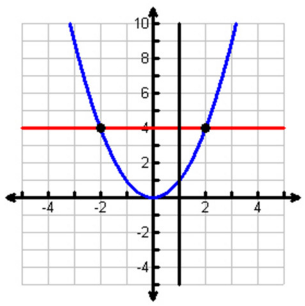
If no horizontal line intersects the graph of a function f more than once, then the inverse of f is itself a function. This is called the **horizontal line test**.

The **inverse of a function** is formed when the independent variable is exchanged with the dependent variable in a given relation. (Switch the x and y with each other.) A function takes a starting value, performs some operation on this value, and creates an output answer. The inverse of a function takes the output answer, performs some operation on it, and arrives back at the original function's starting value. Inverses are indicated by the notation f^{-1}.

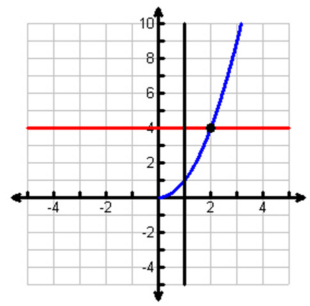
A function is a **one-to-one function** if and only if each second element corresponds to one and only one first element.

In order for the inverse of a function to be a function, the original function must be a one-to-one function and meets the criteria for the vertical and horizontal line tests.

Not all functions meet the criteria to have an inverse which is also a function. However, if the **domain is restricted**, or in other words only part of the domain is used, then the inverse will be a function.



This example is not one-to-one. It is a function because the vertical line intersects the graph only once. However, the horizontal line intersects the graph twice. There is an inverse to this example, but the inverse will not be a function.



This example is one-to-one. It is a function because the vertical and horizontal lines intersect the graph only once. The inverse will be a function.

Example:

Find the inverse of $f(x) = 3x - 1$.

Original function	$f(x) = 3x - 1$
Replace $f(x)$ with y.	$y = 3x - 1$
Replace x with y and y with x.	$x = 3y - 1$
Isolate y.	$x + 1 = 3y$ $\frac{x+1}{3} = y$ $\frac{1}{3}x + \frac{1}{3} = y$

The inverse of $f(x) = 3x - 1$ is $f^{-1}(x) = \frac{1}{3}x + \frac{1}{3}$.

Example:

Find the inverse of $f\left(x\right)=\frac{1}{4}x+2$.

Original function	$f\left(x\right)=\frac{1}{4}x+2$
Replace $f(x)$ with *y*.	$y=\frac{1}{4}x+2$
Replace *x* with *y* and *y* with *x*.	$x=\frac{1}{4}y+2$
Isolate *y*.	$x-2=\frac{1}{4}y$ $4\left(x-2\right)=y$ $4x-8=y$

The inverse of $f\left(x\right)=\frac{1}{4}x+2$ is $f^{-1}\left(x\right)=4x-8$.

Example:

Find the inverse of the function $f\left(x\right)=\frac{1}{4}x^2$; Domain $\left(-\infty,0\right]$ and Range $\left[0,\infty\right)$

Original function	$f\left(x\right)=\frac{1}{4}x^2,\ x\leq 0$
Replace $f(x)$ with *y*.	$y=\frac{1}{4}x^2$
Replace *x* with *y* and *y* with *x*.	$x=\frac{1}{4}y^2$
Isolate *y*.	$4x=y^2$
Simplify the radical (Unit 1 Cluster1:N.RN.2)	$\pm\sqrt{4x}=y$ $\pm 2\sqrt{x}=y$
Determine whether to use the positive or negative answer by referring back to the restricted domain. The domain of the original function is restricted to the negative real numbers including zero, therefore, the range of the inverse function must also be the same. This leads us to choose the negative square root.	$-2\sqrt{x}=f^{-1}(x)$ Domain $\left[0,\infty\right)$ and Range $\left(-\infty,0\right]$ Notice the domain and range have switched from the original function's domain and range.

Example:

Find the inverse of the function $f(x) = 3(x+1)^2 - 5$; Domain $[0, \infty)$ and Range $[-5, \infty)$

Original function	$f(x) = 3(x+1)^2 - 5,\ \ x \geq 0$
Replace $f(x)$ with *y*.	$y = 3(x+1)^2 - 5$
Replace *x* with *y* and *y* with *x*.	$x = 3(y+1)^2 - 5$
Isolate *y*.	$x + 5 = 3(y+1)^2$
Simplify the radical (Unit 1 Cluster1:N.RN.2)	$\frac{x+5}{3} = (y+1)^2$ $\pm\sqrt{\frac{x+5}{3}} = y + 1$ $\pm\sqrt{\frac{x+5}{3}} - 1 = y$
Determine whether to use the positive or negative answer by referring back to the restricted domain. The domain of the original function is restricted to the positive real numbers including zero, therefore, the range of the inverse function must also be the same. This leads us to choose the positive square root.	$\sqrt{\frac{x+5}{3}} - 1 = f^{-1}(x)$ Domain $[-5, \infty)$ and Range $[0, \infty)$ Notice the domain and range have switched from the original function's domain and range.

Practice Exercises F

Find the inverse of the following. State the domain and range of the inverse. For problems 7 – 9 restrict the domain to $x \geq 0$.

1. $f(x) = 3x + 2$
2. $f(x) = -4x + 7$
3. $f(x) = 6x + 5$
4. $f(x) = -\frac{4}{5}x + 1$
5. $f(x) = \frac{2}{3}x - 4$
6. $f(x) = \frac{1}{2}x - 3$
7. $f(x) = 3x^2 - 5$
8. $f(x) = (x+2)^2$
9. $f(x) = (x-7)^2 + 9$

Unit 3

Expressions and Equations

Unit 3 Cluster 1 (A.SSE.2): Interpret the Structure of Expressions

Cluster 1: Interpret the structure of expressions

3.1.2 Recognize functions that are quadratic in nature such as $x^4 - x^2 - 6 = (x^2 - 3)(x^2 + 2)$

VOCABULARY

A quadratic pattern can be found in other types of expressions and equations. If this is the case, we say these expressions, equations, or functions are **quadratic in nature**. Recall the standard form of a quadratic expression is $ax^2 + bx + c$, where *a, b,* and *c* are real numbers and $a \neq 0$.

The following are examples of expressions that are quadratic in nature:

Expression:	Notice:	Rewritten:
$3x^6 + 5x^3 - 12$	$(x^3)^2 = x^6$	$3(x^3)^2 + 5(x^3) - 12$
$2x^6 - 5x^3y^2 - 12y^4$	$(x^3)^2 = x^6$ and $(y^2)^2 = y^4$	$2(x^3)^2 - 5(x^3)(y^2) - 12(y^2)^2$
$9x^{1/2} + 12x^{1/4} + 4$	$(x^{1/4})^2 = x^{1/2}$	$9(x^{1/4})^2 + 12(x^{1/4}) + 4$
$4x - 6\sqrt{x} + 9$	$(\sqrt{x})^2 = x$	$4(\sqrt{x})^2 - 6(\sqrt{x}) + 9$
$(x+1)^4 - 2(x+1)^2 - 15$	$\left[(x+1)^2\right]^2 = (x+1)^4$	$\left[(x+1)^2\right]^2 - 2(x+1)^2 - 15$
$16x^{10} - 25$	$(4x^5)^2 = 16x^{10}$ and $(5)^2 = 25$	$(4x^5)^2 - (5)^2$
$x^4 - y^6$	$(x^2)^2 = x^4$ and $(y^3)^2 = y^6$	$(x^2)^2 - (y^3)^2$

Practice exercise A

Determine if the expression is quadratic in nature.

1. $x^4 + x^2 - 12$
2. $2(2x-3)^2 + (2x-3) - 1$
3. $x^3 - 4x + 4$
4. $x^{1/2} - x^{1/4} - 72$
5. $9x^{2/3} - 12x^{1/3} + 4$
6. $4(x+2)^2 - 1$

FACTORING REVIEW

1. **COMMON TERM** a) What number will go into all of the numbers evenly b) Common variable – use the common variable with the **lowest** power	**EXAMPLE** $3x^3+12x^2-6x$ $3x\left(x^2+4x-2\right)$
2. **DIFFERENCE OF TWO SQUARES** Looks like: $a^2-b^2=(a+b)(a-b)$ a) Terms must be perfect squares b) Must be subtraction c) Powers must be even	**EXAMPLE** $4x^2-25$ $(2x+5)(2x-5)$
3. **PERFECT SQUARE TRINOMIALS** Looks like: $a^2+2ab+b^2=(a+b)^2 \quad or \quad (a+b)(a+b)$ or $a^2-2ab+b^2=(a-b)^2 \quad or \quad (a-b)(a-b)$ a) First and last term must be perfect squares b) Middle term is equal to $2ab$ c) Sign in the parenthesis is the same as the first sign	**EXAMPLE** $9x^2+30x+25$ Does the middle term equal $2ab$? $a=3x \;\; and \;\; b=5 \;\; so \;\; 2(3x)(5)=30x$ Yes it does! Therefore $9x^2+30x+25$ factors to: $(3x+5)^2$ or $(3x+5)(3x+5)$
4. **GROUPING** a) Group terms that have something in common b) Factor out common term in each parenthesis c) Write down what is in the parenthesis, they should be identical d) Then add the "left-overs"	**EXAMPLE** $15xy-21x+10y-14$ $(15xy-21x)+(10y-14)$ $3x(5y-7)+2(5y-7)$ $(5y-7)$ $(5y-7)(3x+2)$
5. FACTOR TRINOMIALS BY GROUPING ax^2+bx+c a) Multiply ***a*** and ***c*** b) Find all the factors of the answer	**EXAMPLE** $6x^2-x-15$ (6)(15) = 90 1 and 90 2 and 45 3 and 30

c) Choose the combination that will either give the sum or difference needed to result in ***b***. In this case the difference d) Rewrite the equation using the combination in place of the middle term e) Now group in order to factor f) Factor out the common term in each parenthesis g) Write down what is in the parenthesis, they should be identical h) Add the "left-overs" to complete the answer	5 and 18 6 and 15 9 and 10 9 and -10 $6x^2+9x-10x-15$ $\left(6x^2+9x\right)-\left(10x+15\right)$ Notice that when factoring out a -1 it changes the sign on ***c*** $3x(2x+3)-5(2x+3)$ $(2x+3)$ $(2x+3)(3x-5)$

The same strategies used to factor quadratic expressions can be used to factor anything that is quadratic in nature. (*For more information on factoring, see the factoring lesson in Unit 2.*)

Expression:	**Rewritten:**	**Factor:**
$3x^6+5x^3-12$	$3\left(x^3\right)^2+5\left(x^3\right)-12$	$\left(3x^3-4\right)\left(x^3+3\right)$
$2x^6-5x^3y^2-12y^4$	$2\left(x^3\right)^2-5\left(x^3\right)\left(y^2\right)-12\left(y^2\right)^2$	$\left(2x^3+3y^2\right)\left(x^3-4y^2\right)$
$9x^{1/2}+12x^{1/4}+4$	$9\left(x^{1/4}\right)^2+12\left(x^{1/4}\right)+4$	$\left(3x^{1/4}+2\right)\left(3x^{1/4}+2\right)$ **or** $\left(3x^{1/4}+2\right)^2$
$4x-12\sqrt{x}+9$	$4\left(\sqrt{x}\right)^2-12\left(\sqrt{x}\right)+9$	$\left(2\sqrt{x}-3\right)\left(2\sqrt{x}-3\right)$ **or** $\left(2\sqrt{x}-3\right)^2$
$16x^{10}-25$	$\left(4x^5\right)^2-(5)^2$	$\left(4x^5+5\right)\left(4x^5-5\right)$
x^4-y^6	$\left(x^2\right)^2-\left(y^3\right)^2$	$\left(x^2+y^3\right)\left(x^2-y^3\right)$

Practice set B
Factor each quadratic in nature expression.

1. $144x^2 - 49y^4$
2. $8x^6 + 2x^3 - 15$
3. $100x^8 - 121y^6$
4. $81x^6 - 4$
5. $2x - \sqrt{x} - 1$
6. $4x^4 - 20x^2 + 25$
7. $9x^{10} - 6x^5y + y^2$
8. $12x^{2/5} - 17x^{1/5} + 6$
9. $3x^{2/3} + 10x^{1/3} + 8$

Sometimes rewriting an expression makes it easier to recognize the quadratic pattern.

Example: $(x+1)^4 - 2(x+1)^2 - 15$	
$u = (x+1)$	You can use a new variable to replace $(x+1)$.
$u^4 - 2u^2 - 15$	Now replace every $(x+1)$ in the expression with *u*. Notice how this is quadratic in nature.
$(u^2 - 5)(u^2 + 3)$	We can use quadratic factoring techniques to factor this expression
$\left[(x+1)^2 - 5\right]\left[(x+1)^2 + 3\right]$	You must remember to replace the *u* with $(x+1)$.

Example: $3x^{1/3} - 8x^{1/6} + 4$	
$u = x^{1/6}$	You can use a new variable to replace $x^{1/6}$.
$3u^2 - 8u + 4$	Now replace every $x^{1/6}$ in the expression with *u*. Notice how this is quadratic in nature.
$(3u - 2)(u - 2)$	We can use quadratic factoring techniques to factor this expression
$(3x^{1/6} - 2)(x^{1/6} - 2)$	You must remember to replace the *u* with $x^{1/6}$.

Practice Exercises C
Identify the "u" in each expression, then factor using "u" substitution. Write the factored form in terms of *x*.

1. $(2x)^4 + (2x)^2 - 6$
2. $4(y-3)^2 + 15(y-3) + 9$
3. $5(2x+5)^8 + 21(2x+5)^4 - 20$
4. $3\left(\sqrt[3]{x+3}\right)^6 + \left(\sqrt[3]{x+3}\right)^3 - 2$
5. $x^{1/2} - 7x^{1/4} + 10$
6. $(x+2)^2 + 11(x+2) - 12$
7. $4x - 25$
8. $\left(\frac{1}{x}\right)^2 + \frac{1}{x} - 2$
9. $4(x^2+1)^4 - 9$

Unit 1 Cluster 3 (N.CN.1 and N.CN.2)
Performing Arithmetic Operations With Complex Numbers

Cluster 3: Performing arithmetic operations with complex numbers

1.3.1 $i^2 = -1$, complex number form *a* + *bi*

1.3.2 Add, subtract, and multiply with complex numbers

VOCABULARY

The **imaginary unit**, i, is defined to be $i = \sqrt{-1}$. Using this definition, it would follow that $i^2 = -1$ because $i^2 = i \cdot i = \sqrt{-1} \cdot \sqrt{-1} = -1$.

The number system can be extended to include the set of **complex numbers**. A complex number written in **standard form** is a number *a* + *bi*, where *a* and *b* are real numbers. If *a* = 0, then the number is called **imaginary**. If *b* = 0 then the number is called real.

Simplifying Radicals with *i*

Extending the number system to include the set of complex numbers allows us to take the square root of negative numbers.

Example:

Simplify $\sqrt{-9}$

$\sqrt{-9}$	
$\sqrt{-1 \cdot 9}$ $\sqrt{-1} \cdot \sqrt{9}$	Rewrite the expression using the properties of radicals.
$i \cdot 3$ $3i$	Remember that $i = \sqrt{-1}$.

Example:

Simplify $\sqrt{-24}$

$\sqrt{-24}$	
$\sqrt{-1 \cdot 4 \cdot 6}$ $\sqrt{-1} \cdot \sqrt{4} \cdot \sqrt{6}$	Rewrite the expression using the properties of radicals.
$i \cdot 2 \cdot \sqrt{6}$ $2i\sqrt{6}$	Remember that $i = \sqrt{-1}$.

Practice Exercises A

Simplify each radical

1. $\sqrt{-25}$	2. $\sqrt{-36}$	3. $\sqrt{-144}$
4. $\sqrt{-98}$	5. $\sqrt{-52}$	6. $\sqrt{-22}$

Performing Arithmetic Operations with Complex Numbers

You can add, subtract, and multiply complex numbers. Similar to the set of real numbers, addition and multiplication of complex numbers is associative and commutative.

Adding and Subtracting Complex Numbers

Example:

Add $(3+2i)+(5-4i)$

$(3+2i)+(5-4i)$	
$3+2i+5-4i$ $3+5+2i-4i$	Remove the parentheses. Group like terms together.
$8-2i$	Combine like terms.

Example:

Subtract $(7-5i)-(-2+6i)$

$(7-5i)-(-2+6i)$	
$7-5i+2-6i$ $7+2-5i-6i$	Distribute the negative and remove the parentheses. Group like terms together.
$9-11i$	Combine like terms.

Example:

Simplify $8-(4-5i)+(2+5i)$

$8-(4-5i)+(2+5i)$	
$8-4+5i+2+5i$ $8-4+2+5i+5i$	Distribute the negative and remove the parentheses. Group like terms together.
$6+10i$	Combine like terms.

Multiplying Complex Numbers

Example:

Multiply $-3(-7+6i)$

$-3(-7+6i)$	
$21-18i$	Distribute the negative three to each term in the parentheses.

Example:

Multiply $4i(2+9i)$

$4i(2+9i)$	
$8i+36i^2$	Distribute the $4i$ to each term in the parentheses.
$8i+36(-1)$ $8i-36$	By definition $i^2=-1$ so substitute -1 in for i^2.
$-36+8i$	Write the complex number in standard form.

Example:

Multiply $(-2+9i)(-3-10i)$

Distributive (FOIL) Method

$(-2+9i)(-3-10i)$

$=-2(-3-10i)+9i(-3-10i)$

$=6+20i-27i-90i^2$

*combine like terms

$=6-7i-90i^2$

*remember that $i^2=-1$

$=6-7i-90(-1)$

$=6-7i+90$

$=96-7i$

Box Method

	-2	$9i$
-3	6	$-27i$
$-10i$	$20i$	$-90i^2$

*combine terms on the diagonals of the unshaded boxes(top right to lower left)

$=6-7i-90i^2$

*remember that $i^2=-1$

$=6-7i-90(-1)$

$=6-7i+90$

$=96-7i$

Vertical Method

$$\begin{array}{r} -2+9i \\ \times \quad -3-10i \\ \hline 20i-90i^2 \\ 6-27i \quad\quad \\ \hline 6-\ 7i-90i^2 \end{array}$$

*remember that $i^2=-1$

$=6-7i-90(-1)$

$=6-7i+90$

$=96-7i$

Example:

Multiply $(5+2i)(5-2i)$

Distributive (FOIL) Method

$(5+2i)(5-2i)$

$=5(5-2i)+2i(5-2i)$

$=25-10i+10i-4i^2$

*combine like terms

$=25-4i^2$

*remember that $i^2=-1$

$=25-4(-1)$

$=25+4$

$=29$

Box Method

	5	$2i$
5	25	$10i$
$-2i$	$-10i$	$-4i^2$

*combine terms on the diagonals of the unshaded boxes(top right to lower left)

$=25-4i^2$

*remember that $i^2=-1$

$=25-4(-1)$

$=25+4$

$=29$

Vertical Method

$$\begin{array}{r} 5+2i \\ \times \quad 5-2i \\ \hline -10i-4i^2 \\ 25+10i \quad\quad \\ \hline 25+\ 0i-4i^2 \end{array}$$

*remember that $i^2=-1$

$=25-4(-1)$

$=25+4$

$=29$

Practice Exercises B

Simplify each expression.

1. $(10-i)+(3-6i)$
2. $(9+6i)-(4-4i)$
3. $4i-(7-5i)+(10-7i)$
4. $7+(-4+8i)-(-10+2i)$
5. $-5(8-2i)$
6. $-11i(-2-9i)$
7. $i(9+15i)$
8. $(-6-7i)(4-12i)$
9. $(-6-7i)(4-12i)$

10 $(1+4i)(12+11i)$

11. $(-10-4i)(-7+5i)$
12. $(-10+2i)(-10-2i)$

Unit 1 Cluster 3 Honors (N.CN.3)

H2.1 Find the conjugate of a complex number; use conjugates to find quotients of complex numbers.

VOCABULARY

The **conjugate** of a complex number is a number in the standard complex form $a+bi$, where the imaginary part bi has the opposite sign of the original, for example $a-bi$ has the opposite sign of $a+bi$. **Conjugate pairs** are any pair of complex numbers that are conjugates of each other such as $3+4i$ and $3-4i$.

The product of conjugate pairs is a positive real number.

$(a+bi)\cdot(a-bi)$	$(3+4i)\cdot(3-4i)$
$a^2-abi+abi-b^2i^2$	$9-12i+12i-16i^2$
$a^2-b^2(-1)$	$9+16$
a^2+b^2	25

This property will be used to divide complex numbers.

Example:

Find the conjugate of the following complex numbers.

a. $4i$ b. $-2+5i$ c. $3-i$ d. $-7-2i$

a. The opposite of $4i$ is $-4i$. The conjugate of $4i$ is $-4i$.	b. The opposite of $5i$ is $-5i$. The conjugate of $-2+5i$ is $-2-5i$.
c. The opposite of $-i$ is i. The conjugate of $3-i$ is $3+i$.	d. The opposite of $-2i$ is $2i$. The conjugate of $-7-2i$ is $-7+2i$.

Practice Exercises A

Find the conjugate of the following complex numbers.

1. $6+6i$ 2. $8-9i$ 3. $-2+3i$ 4. $-1+7i$

Divide by an imaginary number bi

If there is an imaginary number in the denominator of a fraction, then the complex number is not in standard complex form. In order to write it in standard complex form, you must multiply the numerator and the denominator by the conjugate of the denominator. This process removes the imaginary unit from the denominator and replaces it with a real number (the product of conjugate pairs is a positive real number) without changing the value of the complex number. Once this is done, you can write the number in standard complex form by simplifying the fraction.

Example:

Write in standard complex form: $\frac{2}{8i}$

$\frac{2}{8i}\cdot\frac{-8i}{-8i}$ $\frac{-16i}{-64i^2}$	The conjugate of $8i$ is $-8i$. Multiply the numerator and the denominator by the conjugate.
$\frac{-16i}{-64(-1)}$ $\frac{-16i}{64}$	Remember $i^2=-1$.
$\frac{-\cancel{16}\cdot i}{\cancel{16}\cdot 4}=\frac{-i}{4}=-\frac{1}{4}i$	Simplify.

Example:

Write in standard complex form: $\frac{6+8i}{9i}$

$\frac{6+8i}{9i}\cdot\frac{-9i}{-9i}$ $\frac{-54i-72i^2}{-81i^2}$	The conjugate of $9i$ is $-9i$. Multiply the numerator and the denominator by the conjugate.
$\frac{-54i-72(-1)}{-81(-1)}$	Remember $i^2=-1$.
$\frac{72-54i}{81}$	Rewrite numerator in standard complex form $a+bi$.
$\frac{72}{81}-\frac{54i}{81}$ $\frac{8}{9}-\frac{2}{3}i$	Rewrite whole solution in complex form $a+bi$, reducing as needed.

Practice Exercises B

Write in standard complex form.

1. $\dfrac{3}{5i}$	2. $\dfrac{6}{-4i}$	3. $\dfrac{-5}{-5i}$
4. $\dfrac{-3+10i}{-6i}$	5. $\dfrac{10-10i}{-5i}$	6. $\dfrac{2-3i}{4i}$

Dividing Complex Numbers in Standard Form $a+bi$

To divide complex numbers, find the complex conjugate of the denominator, multiply the numerator and denominator by that conjugate, and simplify.

Example:

Divide $\dfrac{10}{2+i}$

$\dfrac{10}{2+i}$	
$\dfrac{10}{2+i}\cdot\dfrac{2-i}{2-i}$ $\dfrac{10(2-i)}{(2+i)(2-i)}$	Multiply the numerator and denominator by the conjugate of 2+*i*, which is 2 – *i*
$\dfrac{20-10i}{4-2i+2i-i^2}$	Distribute
$\dfrac{20-10i}{4-i^2}$	Simplify
$\dfrac{20-10i}{4-(-1)}$ $\dfrac{20-10i}{5}$	Note that $i^2=-1$
$\dfrac{20}{5}-\dfrac{10}{5}i=4-2i$	Simplify and write in standard complex form.

Example:

Divide $\frac{22-7i}{4-5i}$

$\frac{22-7i}{4-5i}$	
$\frac{22-7i}{4-5i}\cdot\frac{4+5i}{4+5i}$ $\frac{(22-7i)(4+5i)}{(4-5i)(4+5i)}$	Multiply by the conjugate of the denominator.
$\frac{88+110i-28i-35i^2}{16+20i-20i-25i^2}$	Distribute
$\frac{88+82i-35i^2}{16-25i^2}$ $\frac{88+82i-35(-1)}{16-25(-1)}$ $\frac{88+82i+35}{16+25}$ $\frac{123+82i}{41}$	Combine like terms. Remember that $i^2=-1$. Combine like terms again.
$\frac{123}{41}+\frac{82}{41}i$ $3+2i$	Simplify and write in standard complex form.

Example:

Divide $\frac{6+2i}{1-2i}$

$\frac{6+2i}{1-2i}$	
$\frac{6+2i}{1-2i}\cdot\frac{1+2i}{1+2i}$	Multiply by the conjugate of the denominator.

$\dfrac{(6+2i)(1+2i)}{(1-2i)(1+2i)}$	
$\dfrac{6+12i+2i+4i^2}{1+2i-2i-4i^2}$	Distribute.
$\dfrac{6+14i+4i^2}{1-4i^2}$ $\dfrac{6+14i+4(-1)}{1-4(-1)}$ $\dfrac{6+14i-4}{1+4}$ $\dfrac{2+14i}{5}$	Combine like terms. Remember $i^2=-1$ Combine like terms again.
$\dfrac{2}{5}+\dfrac{14}{5}i$	Put in standard complex form $a+bi$.

Practice Exercises C

Divide each complex rational expression and write in standard complex form.

1. $\dfrac{5i}{-2-6i}$
2. $\dfrac{8i}{-1+3i}$
3. $\dfrac{10}{-3-i}$
4. $\dfrac{26+18i}{3+4i}$
5. $\dfrac{-10-5i}{-6+6i}$
6. $\dfrac{-3-7i}{7+10i}$

Unit 3 Cluster 4 (A.REI.4) and Unit 3 Cluster 5 (N.CN.7): Solve Equations and Inequalities in One Variable

Cluster 4: Solving equations in one variable

- 3.4.1a Derive the quadratic formula by completing the square.
- 3.4.1b Solve equations by taking the square root, completing the square, using the quadratic formula and by factoring (recognize when the quadratic formula gives complex solutions and write them as $a \pm bi$)
- 3.5.1 Solve with complex numbers

VOCABULARY

The **square root** of a number is a value that, when multiplied by itself, gives the number. For example if $r^2 = a$, then *r* is the square root of *a*. There are two possible values for *r*; one positive and one negative. For instance, the square root of 9 could be 3 because $3^2 = 9$ but it could also be -3 because $(-3)^2 = 9$.

A **perfect square** is a number that can be expressed as the *product of two equal integers*. For example: 100 is a perfect square because $10 \cdot 10 = 100$ and x^2 is a perfect square because $x \cdot x = x^2$.

Solving Equations by Taking the Square Root

When solving a quadratic equation by taking the square root, you want to isolate the squared term so that you can take the square root of both sides of the equation.

Example:

Solve the quadratic equation $x^2 = 4$.

$x^2 = 4$	In this example the squared term is x^2 and it is already isolated.
$\sqrt{x^2} = \sqrt{4}$	Take the square root of each side of the equation.
$\left(x^2\right)^{1/2} = \pm\sqrt{4}$ $x^{2/2} = \pm 2$ $x = \pm 2$ $x = 2$ or $x = -2$	Using the properties of rational exponents you can simplify the left side of the equation to *x*. The number 16 is a perfect square because $2 \cdot 2 = 4$.

Example:

Solve the quadratic equation $\frac{2}{3}x^2+4=16$.

$\frac{2}{3}x^2+4=16$ $\frac{2}{3}x^2=12$ $x^2=18$	In this example the squared term is x^2 and it needs to be isolated. Use a reverse order of operations to isolate x^2.
$\sqrt{x^2}=\sqrt{18}$	Take the square root of each side of the equation.
$\left(x^2\right)^{1/2}=\pm\sqrt{9\cdot 2}$ $x^{2/2}=\pm\sqrt{9}\cdot\sqrt{2}$ $x=\pm 3\sqrt{2}$ $x=3\sqrt{2}$ or $x=-3\sqrt{2}$	Using the properties of rational exponents you can simplify the left side of the equation to *x*. Using the properties of radical expressions you can simplify the right side of the equation. (*See Unit 1 Cluster 2 for help with simplifying*)

Example:

Solve the quadratic equation $3(x-2)^2+4=52$.

$3(x-2)^2+4=52$ $3(x-2)^2=48$ $(x-2)^2=16$	In this example the squared term is $(x-2)^2$ and it needs to be isolated. Use a reverse order of operations to isolate $(x-2)^2$.
$\sqrt{(x-2)^2}=\sqrt{16}$	Take the square root of each side of the equation.
$\left[(x-2)^2\right]^{1/2}=\pm\sqrt{16}$ $(x-2)^{2/2}=\pm 4$ $x-2=\pm 4$ $x-2=6$ or $x-2=-6$	Using the properties of rational exponents you can simplify the left side of the equation to $x-2$. The number 16 is a perfect square because $4\cdot 4=16$.
$x-2=6$ or $x-2=-6$ $x=8$ or $x=-4$	You still need to solve each equation for *x*.

Practice Exercises A
Solve each quadratic equation.

1. $x^2 = 25$
2. $x^2 = 8$
3. $-4x^2 = -36$
4. $4x^2 - 5 = -1$
5. $9x^2 - 3 = 33$
6. $16 - x^2 = -9$
7. $-2(x-3)^2 + 4 = -8$
8. $3(x+3)^2 - 1 = 2$
9. $\frac{1}{3}(x+1)^2 = 5$

Solving Quadratic Equations by Completing the Square

Sometimes you have to rewrite a quadratic equation, using the method of completing the square, so that it can be solved by taking the square root.

Example:

Solve $x^2 - 10x = -23$.

$x^2 - 10x = -23$	
$x^2 - 10x + \left(\frac{10}{2}\right)^2 = -23 + \left(\frac{10}{2}\right)^2$ $x^2 - 10x + 25 = -23 + 25$ $x^2 - 10x + 25 = 2$	Complete the square on the left side of the equation.
$(x-5)(x-5) = 2$ $(x-5)^2 = 2$	Factor the expression on the left side.
$\sqrt{(x-5)^2} = \sqrt{2}$	Take the square root of each side.
$x - 5 = \pm\sqrt{2}$	Simplify.
$x - 5 = \sqrt{2}$ or $x - 5 = -\sqrt{2}$ $x = 5 + \sqrt{2}$ or $x = 5 - \sqrt{2}$	Solve for *x*.

Example:

Solve $x^2+4x=-6$.

$x^2+4x=-5$	
$x^2+4x+\left(\frac{4}{2}\right)^2=-5+\left(\frac{4}{2}\right)^2$ $x^2+4x+4=-5+4$ $x^2+4x+4=-1$	Complete the square on the left side of the equation.
$(x+2)(x+2)=-1$ $(x+2)^2=-1$	Factor the expression on the left side.
$\sqrt{(x+2)^2}=\sqrt{-1}$	Take the square root of each side.
$x+2=\pm i$	Simplify. Remember that the $\sqrt{-1}=i$
$x+2=i$ or $x+2=-i$ $x=-2+i$ or $x=-2-i$	Solve for *x*.

Example:

Solve $3x^2-5x-2=0$.

$3x^2-5x-2=0$ $3x^2-5x=2$	Collect the terms with variables on one side of the equation and the constant term on the other side.
$3\left(x^2-\frac{5}{3}x\right)=2$ $3\left[\left(x^2-\frac{5}{3}x\right)+\left(\frac{5}{2\cdot 3}\right)^2\right]=2+3\cdot\left(\frac{5}{2\cdot 3}\right)^2$ $3\left(x^2-\frac{5}{3}x+\frac{25}{36}\right)=2+3\cdot\frac{25}{36}$ $3\left(x^2-\frac{5}{3}x+\frac{25}{36}\right)=2+\frac{75}{36}$ $3\left(x^2-\frac{5}{3}x+\frac{25}{36}\right)=\frac{49}{12}$	Complete the square on the left side of the equation.

$3\left(x-\frac{5}{6}\right)\left(x-\frac{5}{6}\right)=\frac{49}{12}$ $3\left(x-\frac{5}{6}\right)^2=\frac{49}{12}$ $\left(x-\frac{5}{6}\right)^2=\frac{49}{36}$	Factor the expression on the left side of the equation. Isolate the squared term.
$\sqrt{\left(x-\frac{5}{6}\right)^2}=\sqrt{\frac{49}{36}}$	Take the square root of each side.
$x-\frac{5}{6}=\pm\frac{7}{6}$	Simplify.
$x-\frac{5}{6}=\frac{7}{6}$ or $x-\frac{5}{6}=-\frac{7}{6}$ $x=\frac{5}{6}+\frac{7}{6}$ or $x=\frac{5}{6}-\frac{7}{6}$ $x=2$ or $x=-\frac{1}{3}$	Solve for *x*.

Practice Exercises B
Solve the quadratic equations by completing the square.

1. $x^2-4x=-1$
2. $x^2+12x=-32$
3. $x^2+16x+15=0$
4. $x^2+8x-3=0$
5. $x^2+10=6x$
6. $2x^2+4x-5=0$

VOCABULARY

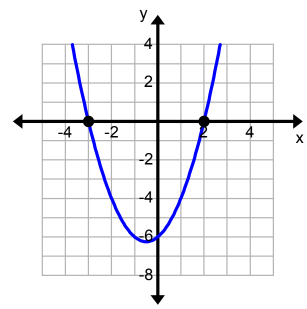
The **quadratic formula**, $x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$, can be used to find the solutions of the quadratic equation $ax^2+bx+c=0$, when $a\neq 0$. The portion of the quadratic equation that is under the radical, b^2-4ac, is called the **discriminant**. It can be used to determine the number and type of solutions to the quadratic equation $ax^2+bx+c=0$.

Using the Determinant to Determine Number and Type of Solutions

- If $b^2 - 4ac > 0$, then there are two real solutions to the quadratic equation.

Example:

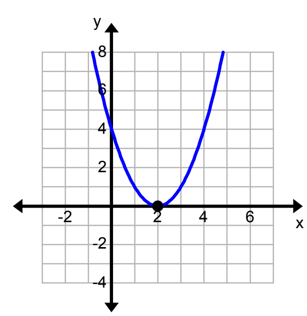
Determine the number and type of solutions for the equation $x^2 + x - 6 = 0$.

<table>
<tr><td>$x^2 + x - 6 = 0$
$a = 1,\ b = 1,$ and $c = -6$</td><td>Identify a, b, and c.</td><td rowspan="5">
</td></tr>
<tr><td>$b^2 - 4ac$
$1^2 - 4(1)(-6)$</td><td>Substitute the values of a, b, and c into the discriminant formula.</td></tr>
<tr><td>$1 - 4(-6)$
$1 - (-24)$
$1 + 24$
25</td><td>Simplify using order of operations.</td></tr>
<tr><td>$25 > 0$</td><td>Determine if the result is greater than zero, equal to zero, or less than zero.</td></tr>
<tr></tr>
<tr><td colspan="3">The quadratic equation $x^2 + x - 6 = 0$ has two real solutions. You can see from the graph that the function crosses the x-axis twice.</td></tr>
</table>

- If $b^2 - 4ac = 0$, then there is one real solution to the quadratic equation.

Example:

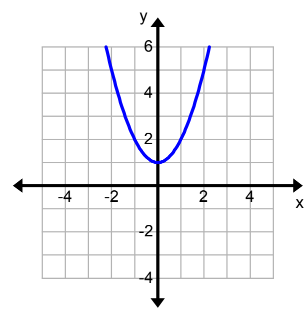
Determine the number and type of solutions for the equation $x^2 + 4x + 4 = 0$.

<table>
<tr><td>$x^2 + 4x + 4 = 0$
$a = 1,\ b = 4,$ and $c = 4$</td><td>Identify a, b, and c.</td><td rowspan="5">
</td></tr>
<tr><td>$b^2 - 4ac$
$4^2 - 4(1)(4)$</td><td>Substitute the values of a, b, and c into the discriminant formula.</td></tr>
<tr><td>$16 - 4(4)$
$16 - 16$
0</td><td>Simplify using order of operations.</td></tr>
<tr><td>$0 = 0$</td><td>Determine if the result is greater than zero, equal to zero, or less than zero.</td></tr>
<tr></tr>
<tr><td colspan="3">The quadratic equation $x^2 + 4x + 4 = 0$ has one real solution. You can see from the graph that the function touches the x-axis only once.</td></tr>
</table>

- If $b^2 - 4ac < 0$, then there no real, but two imaginary solutions to the quadratic equation.

Example:

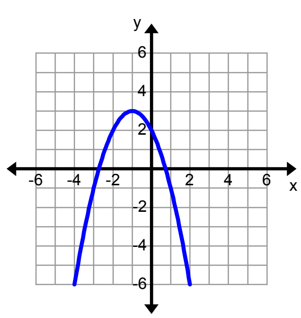
Determine the number and type of solutions for the equation $x^2 + 1 = 0$.

$x^2 + 1 = 0$ $a = 1,\ b = 0$, and $c = 1$ $b^2 - 4ac$ $0^2 - 4(1)(1)$ $0 - 4(1)$ $0 - 4$ -4 $-4 < 0$	Identify *a*, *b*, and *c*. Substitute the values of *a*, *b*, and *c* into the discriminant formula. Simplify using order of operations. Determine if the result is greater than zero, equal to zero, or less than zero.	
The quadratic equation $x^2 + 1 = 0$ has no real solutions, but it has two imaginary solutions. You can see from the graph that the function never crosses the *x*-axis.		

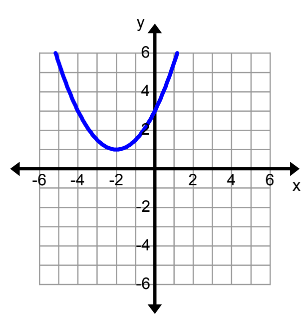
Practice Exercises C

Determine the number and type of solutions that each quadratic equation has.

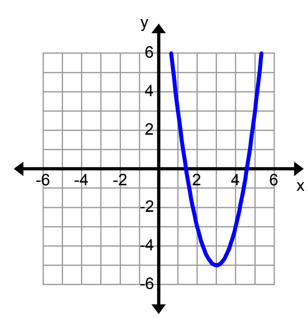
1.



2.



3.



4. $-x^2 + 2x - 5 = 0$

5. $4x^2 + 12x + 9 = 0$

6. $2x^2 + 4x = 2$

Solving Quadratic Equations by Using the Quadratic Formula

Example:

Solve $3x^2 + 5x - 4 = 0$ using the quadratic formula.

$3x^2 + 5x - 4 = 0$ $a = 3,\ b = 5,\ \text{and } c = -4$	Make sure all the terms are on the same side and that the equation equals 0. Identify *a*, *b*, and *c*.
$x = \dfrac{-5 \pm \sqrt{5^2 - 4(3)(-4)}}{2(3)}$	Substitute the values for *a*, *b*, and *c* into the quadratic formula.
$x = \dfrac{-5 \pm \sqrt{25 - 12(-4)}}{6}$ $x = \dfrac{-5 \pm \sqrt{25 - (-48)}}{6}$ $x = \dfrac{-5 \pm \sqrt{25 + 48}}{6}$ $x = \dfrac{-5 \pm \sqrt{73}}{6}$	Use order of operations to simplify.
$x = \dfrac{-5 + \sqrt{73}}{6}$ and $x = \dfrac{-5 - \sqrt{73}}{6}$	These are actually two different solutions.

Example:

Solve $2x^2 - x - 3 = 0$ using the quadratic formula.

$2x^2 - x - 3 = 0$ $a = 2,\ b = -1,\ \text{and } c = -3$	Make sure all the terms are on the same side and that the equation equals 0. Identify *a*, *b*, and *c*.
$x = \dfrac{1 \pm \sqrt{(-1)^2 - 4(2)(-3)}}{2(2)}$	Substitute the values for *a*, *b*, and *c* into the quadratic formula.

$x=\dfrac{1\pm\sqrt{1-8(-3)}}{4}$ $x=\dfrac{1\pm\sqrt{1-(-24)}}{4}$ $x=\dfrac{1\pm\sqrt{1+24}}{4}$ $x=\dfrac{1\pm\sqrt{25}}{4}$ $x=\dfrac{1\pm 5}{4}$	Use order of operations to simplify.
$x=\dfrac{1+5}{4}=\dfrac{6}{4}=\dfrac{3}{2}$ and $x=\dfrac{1-5}{4}=\dfrac{-4}{4}=-1$	Simplify each answer.

Example:

Solve $25x^2+10x=-1$ using the quadratic formula.

$25x^2+10x=-1$ $25x^2+10x+1=0$ $a=25,\ b=10,$ and $c=1$	Make sure all the terms are on the same side and that the equation equals 0. Identify *a*, *b*, and *c*.
$x=\dfrac{-10\pm\sqrt{10^2-4(25)(1)}}{2(25)}$	Substitute the values for *a*, *b*, and *c* into the quadratic formula.
$x=\dfrac{-10\pm\sqrt{100-100(1)}}{50}$ $x=\dfrac{-10\pm\sqrt{100-100}}{50}$ $x=\dfrac{-10\pm\sqrt{0}}{50}$ $x=\dfrac{-10\pm 0}{50}$ $x=\dfrac{-10}{50}$	Use order of operations to simplify.
$x=-\dfrac{1}{5}$	Simplify the answer. Notice that we only got one answer this time because the discriminant was 0.

Example:

Solve $2x^2+12x=-20$ using the quadratic formula.

$2x^2+12x=-20$ $2x^2+12x+20=0$ $a=2,\ b=12,\ \text{and } c=20$	Make sure all the terms are on the same side and that the equation equals 0. Identify *a*, *b*, and *c*.
$x=\dfrac{-12\pm\sqrt{12^2-4(2)(20)}}{2(2)}$	Substitute the values for *a*, *b*, and *c* into the quadratic formula.
$x=\dfrac{-12\pm\sqrt{144-8(20)}}{4}$ $x=\dfrac{-12\pm\sqrt{144-160}}{4}$ $x=\dfrac{-12\pm\sqrt{-16}}{4}$ $x=\dfrac{-12\pm 4i}{4}$ $x=-3\pm i$	Use order of operations to simplify.
$x=-3+i$ or $x=-3-i$	Simplify the answer. Notice that we got two imaginary answers this time because the discriminant was less than 0.

Practice Exercises D

Solve the quadratic equation using the quadratic formula.

1. $-3x^2-5x+7=0$
2. $4x^2+12x+9=0$
3. $6x^2+11x-7=0$
4. $x^2+8x-12=0$
5. $x^2+3=6x$
6. $x^2-2x=2$
7. $x^2-x+1=0$
8. $2x^2+5x+4=0$
9. $-3x^2-4=-2x$

Practice Exercises E

Solve each of the following equations using the method of your choice.

1. $2x^2 - 5x + 4 = 0$
2. $x^2 + 10 = 6x$
3. $x^2 + 4x + 6 = 0$
4. $x(x-3) = x - 9$
5. $x(x-1) = 2x - 7$
6. $x^2 + 10x + 26 = 0$
7. $4x^2 + 81 = 0$
8. $(x+1)^2 = -9$
9. $(x-1)^2 + 5 = -44$
10. $x^2 + x + 4 = 0$
11. $x^2 + 4 = 0$
12. $2(x+3)^2 - 5 = -23$

Unit 3 Cluster 5 (N.CN.8, N.CN.9-Honors):
Use complex numbers in polynomial identities and equations

3.5.2 Extend polynomial identities to the complex numbers.
3.5.3 Know the Fundamental Theorem of algebra; show that it is true for quadratic polynomials.

The **Fundamental Theorem of Algebra** states that every polynomial of degree *n* with complex coefficients has exactly *n* roots in the complex numbers.

Note: Remember that every root can be written as a complex number in the form of $a+bi$. For instance $x=3$ can be written as $x=3+0i$. In addition, all complex numbers come in conjugate pairs, $a+bi$ and $a-bi$.

Example:

$f(x)=x^2+x+3$	Degree: 2 Complex Roots: 2
$f(x)=5x^3+2x^2-5x+4$	Degree: 3 Complex Roots: 3

Polynomial Identities

1. $(a+b)^2=a^2+2ab+b^2$
2. $(a+b)(c+d)=ac+ad+bc+bd$
3. $a^2-b^2=(a+b)(a-b)$
4. $x^2+(a+b)x+AB=(x+a)(x+b)$
5. If $ax^2+bx+c=0$ then $x=\dfrac{-b\pm\sqrt{b^2-4ac}}{2a}$

Example:
Find the complex roots of $f(x)=x^2+64$ and write in factored form.

$0=x^2+64$ $-64=x^2$ $\sqrt{-64}=\sqrt{x^2}$ $\pm 8i=x$	1. Set equal to zero to find the roots of the function. Solve.
$(x-8i)(x-(-8i))=f(x)$ $(x-8i)(x+8i)=f(x)$	2. Recall factored form is $(x-p)(x-q)=f(x)$. Substitute the zeros in for *p* and *q*.

Example:
Find the complex roots of $f(x)=x^2+16x+65$ and write in factored form.

$x=\dfrac{-16\pm\sqrt{(16)^2-4(1)(65)}}{2(1)}$ $x=\dfrac{-16\pm\sqrt{256-260}}{2}$ $x=\dfrac{-16\pm\sqrt{-4}}{2}$ $x=\dfrac{-16\pm 2i}{2}$ $x=-8\pm i$	1. Use the quadratic formula to find the roots of the function.
$(x-(-8+i))(x-(-8-i))=f(x)$ $(x+8-i)(x+8+i)=f(x)$	2. Recall factored form is $(x-p)(x-q)=f(x)$. Substitute the zeros in for *p* and *q*.

Example:
Find the complex roots of $f(x)=x^4+10x^2+24$ and write in factored form.

$f(x)=\left(x^2+4\right)\left(x^2+6\right)$	1. Factor the quadratic in nature function.
$x^2+4=0$ $\quad$ $x^2+6=0$ $x^2=-4$ $\quad$ $x^2=-6$ $x=\pm 2i$ $\quad$ $x=\pm i\sqrt{6}$	2. Set each factor equal to zero to find the roots.
$f(x)=(x-2i)(x-(-2i))\left(x-i\sqrt{6}\right)\left(x-\left(-i\sqrt{6}\right)\right)$ $f(x)=(x-2i)(x+2i)\left(x-i\sqrt{6}\right)\left(x+i\sqrt{6}\right)$	3. Recall factored form is $(x-p)(x-q)=f(x)$. Substitute the zeros in for *p* and *q*.

Practice Exercises A
Find the complex roots. Write in factored form.

1. x^2+9
2. x^2+x+1
3. x^2-2x+2
4. $x^2-6x+10$
5. x^2-4x+5
6. x^2-2x+5
7. x^4+5x^2+4
8. x^4+13x^2+36
9. x^4-1

Unit 3 Cluster 3 (A.CED.1, A.CED.4)
Writing and Solving Equations and Inequalities

Cluster 3: Creating equations that describe numbers or relationships

- 3.3.1 Write and solve equations and inequalities in one variable (including linear, simple exponential, and quadratic functions)
- 3.3.3 Solve formulas for a variable including those involving squared variables

Writing and Solving Quadratic Equations in One Variable

When solving contextual type problems it is important to:

- Identify what you know.
- Determine what you are trying to find.
- Draw a picture to help you visualize the situation when possible. Remember to label all parts of your drawing.
- Use familiar formulas to help you write equations.
- Check your answer for reasonableness and accuracy.
- Make sure you answered the entire question.
- Use appropriate units.

Example:

Find three consecutive integers such that the product of the first two plus the square of the third is equal to 137.

First term: x Second term: $x + 1$ Third term: $x + 2$	The first number is x. Since they are consecutive numbers, the second term is one more than the first or $x + 1$. The third term is one more than the second term or $x+1+1=x+2$.
$x(x+1)+(x+2)^2=137$	Multiply the first two together and add the result to the third term squared. This is equal to 137.
$x(x+1)+(x+2)(x+2)=137$ $x^2+x+x^2+4x+4=137$ $2x^2+5x+4=137$	Multiply and combine like terms.
$2x^2+5x-133=0$	Make sure the equation is equal to 0.
$2x^2-14x+19x-133=0$ $\left(2x^2-14x\right)+\left(19x-133\right)=0$ $2x\left(x-7\right)+19\left(x-7\right)=0$ $\left(x-7\right)\left(2x+19\right)=0$	Factor.

$x-7=0$ $x=7$ $2x+19=0$ $2x=-19$ $x=-\frac{19}{2}$	Use the Zero Product Property to solve for *x*.
First term: 7 Second term: 8 Third term: 9	The numbers are integers so *x* has to be 7.

Example:

A photo is 6 in longer than it is wide. Find the length and width if the area is 187 in^2.

$\text{width} = x$ $\text{length} = x+6$ $A(x) = wl = x(x+6)$ $187 = x(x+6)$	The width is the basic unit, so let it equal *x*. The length is 6 inches longer than width or $x+6$. A photo is rectangular so the area is equal to the width times the length. The area is 187 square inches.
$187 = x^2+6x$	Multiply the right side.
$0 = x^2+6x-187$	Make sure the equation equals 0.
$0=(x-11)(x+17)$	Factor the expression on the right side of the equation.
$x-11=0$ $x=11$ $x+17=0$ $x=-17$	Use the Zero Product Property to solve for *x*.
The width is 11 inches and the length is 17 inches.	The length of a photo cannot be negative. Therefore, *x* must be 11. The length is $x+6=11+6=17$.

Note: Often problems will require information from more than one equation to solve. For example, you might need the perimeter equation to help you write the area equation or vice versa. The **primary** equation is the equation you solve to find the answer you are looking for. The **secondary** equation is the equation you use to help set up your primary equation.

Example:

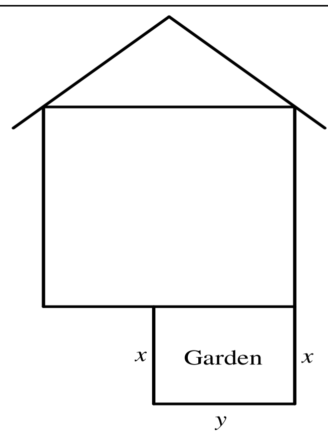
Find two numbers that add to 150 and have a maximum product. What is the maximum product?

Secondary equation: $x+y=150$	One number is *x*. The other number is *y*. The sum is 150. Write an equation for this.

$y = 150 - x$	Use the sum equation to solve for y.
Primary Equation: $P(x) = x(150 - x)$	Write an equation for the product in terms of x.
$P(x) = 150x - x^2$	Simplify the right side of the equation.
$P(x) = -x^2 + 150x$ $P(x) = -(x^2 - 150x)$ $P(x) = -(x^2 - 150x + 5625) + 5625$ $P(x) = -(x - 75)^2 + 5625$	Find the maximum. Remember the maximum is the vertex. Using the method of your choice from Unit 2 Lesson F.IF.8.
$x = -\dfrac{150}{2(-1)}$ $x = 75$ $P(x) = 150x - x^2$ $P(75) = 150(75) - (75)^2$ $P(75) = 11250 - 5625$ $P(75) = 5625$	
$x = 75$ $y = 75$ The two numbers are 75 and 75	The second number is $150 - x$ or 75.
$75 \cdot 75 = 5625$ The maximum product is 5625.	Find the maximum product.

Example:

Jason wants to fence in a rectangular garden in his backyard. If one side of the garden is against the house and Jason has 48 feet of fencing, what dimensions will maximize the garden area while utilizing all of the fencing?

	First draw a picture of the house and garden. Label the sides of your garden. The amount of fence used is the distance around the garden excluding the side next to the house. This is the same as the perimeter.

$2x + y = 48$	The length of the garden is *x* and the width is *y*. The perimeter is 48. Write an equation for this.
$y = 48 - 2x$	Use the sum equation to solve for *y*.
$A(x) = lw$ $A(x) = x(48 - 2x)$	Write an equation for the product in terms of *x*.
$P(x) = 48x - 2x^2$	Simplify the right side of the equation.
$A(x) = -2x^2 + 48x$ $A(x) = -2(x^2 - 24x)$ $A(x) = -2(x^2 - 24x + 144) + 288$ $A(x) = -2(x - 12)^2 + 288$	Find the maximum area. Remember the maximum is the vertex. Using the method of your choice from Unit 2 Lesson F.IF.8.
$x = -\dfrac{48}{2(-2)}$ $x = 12$ $A(x) = 48x - 2x^2$ $P(12) = 48(12) - 2(12)^2$ $P(12) = 576 - 288$ $P(12) = 288$	
$x = 12$ $y = 24$ The length is 12 feet and the width is 24 feet.	The second number is $48 - 2(12)$ or 24.
$12 \cdot 24 = 288$ The maximum area is 288 ft^2.	Find the maximum area.

Practice Problems A

Solve

1. The maximum size envelope that can be mailed with a large envelope rate is 3 inches longer than it is wide. The area is 180 in^2. Find the length and the width.

2. A rectangular garden is 30 ft. by 40 ft. Part of the garden is removed in order to install a walkway of uniform width around it. The area of the new garden is one-half the area of the old garden. How wide is the walkway?

3. The base of a triangular tabletop is 20 inches longer than the height. The area is 750 in^2. Find the height and the base.

4. Find two numbers that differ by 8 and have a minimum product.

5. Alec has written an award winning short story. His mother wants to frame it with a uniform border. She wants the finished product to have an area of 315 in^2. The writing portion occupies an area that is 11 inches wide and 17 inches long. How wide is the border?

6. Britton wants to build a pen for his teacup pig. The length must be 5 feet longer than the width. The pig doesn't need a lot of room so the area should be minimized so it doesn't take up too much of the yard. What should the dimensions of the pen be? What is the minimum area?

7. The product of 2 numbers is 476. One number is 6 more than twice the first number. Find the two numbers.

8. Find three consecutive integers such that the square of the second number plus the product of the first and third numbers is a minimum.

VOCABULARY

Objects that are shot, thrown, or dropped into the air are called **projectiles**. Their height, measured from the ground, can be modeled by a **projectile motion equation**. The object is always affected by gravity. The gravitational constant is different depending on the units of measurement. For example, the gravitational constant in feet is -32 ft/sec^2 and in meters it is -9.8 m/sec^2. Similarly, the projectile motion equation for an object shot or thrown straight up or down is different depending on the units of measurement.

$$\text{Feet: } h(t) = -16t^2 + v_0 t + h_0$$

$$\text{Meters: } h(t) = -4.9t^2 + v_0 t + h_0$$

$h(t)$ represents the height at any time *t*. The time is measured in seconds. The **initial velocity**, v_0, is the speed at which the object is thrown or shot. It is measured in ft/sec or m/sec. The **initial height**, h_0, is the height that the object is shot or thrown from. It is measured in feet or meters.

Example:

The Willis Tower (formerly Sears Tower) in Chicago, Illinois is the tallest building in the United States. It is 108 stories or about 1,451 feet high. Alain Robert, also known as Spiderman, attempted to climb the building in 1999, but he was unable to make it to the top. (Assume that each floor is 13 feet high.)

a. If Spiderman is 28 floors from the top and he drops a piece of equipment, how long will it take for the equipment to reach the ground?

b. How far from the ground is the piece of equipment after 5 seconds?

c. When does the equipment pass the 16th floor?

a. $h(t) = -16t^2 + v_0 t + h_0$	The building is measured in feet so use the projectile motion equation for feet.
$h(t) = -16t^2 + (0)t + (108 - 28) \cdot 13$ $h(t) = -16t^2 + 80 \cdot 13$ $h(t) = -16t^2 + 1040$	The equipment was dropped making the initial velocity 0 ft./sec. The building has 108 floors, but he stopped 28 short of the top floor. Each floor is 13 feet high; multiply the number of floors by the height of each floor to get the initial height.
$0 = -16t^2 + 1040$	We want to know when the equipment hits the ground making the final height zero.
$0 = -16t^2 + 1040$ $16t^2 = 1040$ $t^2 = 65$ $t = \pm\sqrt{65}$	Solve for *t*.
$t = \sqrt{65}$ sec. or 8.062 sec.	Negative time means you are going back in time. Therefore, time is positive.

b. $h(t) = -16t^2 + 1040$ $h(5) = -16(5)^2 + 1040$ $h(5) = -400 + 1040$ $h(5) = 640$	We want to know when the height of the equipment at 5 seconds.
The equipment is 640 feet from the ground after 5 seconds.	

c. $h(t) = -16t^2 + 1040$	We want to know when the equipment passes the 16^{th} floor. The equation is the same equation written in part a.
$16 \cdot 13 = -16t^2 + 1040$ $208 = -16t^2 + 1040$	The 16^{th} floor is 208 feet above the ground.
$0 = -16t^2 + 832$ $-832 = -16t^2$ $52 = t^2$ $\sqrt{52} = \sqrt{t^2}$ $\pm 2\sqrt{13} = t$	Solve for *t*.
$t = 2\sqrt{13}$ sec. or 7.211 sec.	Negative time means you are going back in time. Therefore, time is positive.

Example:

The Salt Lake Bees are planning to have a fireworks display after their game with the Tacoma Rainiers. Their launch platform is 5 feet off the ground and the fireworks will be launched with an initial of 30 feet per second. How long will it take each firework to reach their maximum height?

<table>
<tr><td>a.
$h(t) = -16t^2 + v_0 t + h_0$</td><td>The height of the fireworks is measured in feet so use the projectile motion equation for feet.</td></tr>
<tr><td>$h(t) = -16t^2 + (32)t + 5$
$h(t) = -16t^2 + 32t + 5$</td><td>The fireworks were launched with an initial velocity 30 ft./sec. The launch platform is 5 feet off the ground.</td></tr>
<tr><td>$h(t) = -16t^2 + 32t + 5$
$h(t) = -16\left(t^2 - 2t\right) + 5$
$h(t) = -16\left(t^2 - 2t + 1\right) + 5 + 16$
$h(t) = -16\left(t - 1\right)^2 + 21$</td><td rowspan="2">Using the method of your choice from Unit 2 Lesson F.IF.8. Find the amount of time it will take to reach the maximum height. The t coordinate of the vertex indicates WHEN the firework will reach its maximum height.</td></tr>
<tr><td>$t = -\dfrac{b}{2a}$
$t = -\dfrac{32}{2(-16)} = 1$</td></tr>
<tr><td colspan="2">The firework will reach its maximum height after 1 second.</td></tr>
</table>

Practice Exercises B

Solve.

1. A bolt falls off an airplane at an altitude of 500 m. How long will it take the bolt to reach the ground?
2. A ball is thrown upward at a speed of 30 m/sec from an altitude of 20 m. What is the maximum height of the ball?
3. How far will an object fall in 5 seconds if it is thrown downward at an initial velocity of 30 m/sec from a height of 200 m?
4. A ring is dropped from a helicopter at an altitude of 246 feet. How long does it take the ring to reach the ground?
5. A coin is tossed upward with an initial velocity of 30 ft/sec from an altitude of 8 feet. What is the maximum height of the coin?
6. What is the height of an object after two seconds, if thrown downward at an initial velocity of 20 ft/sec from a height of 175 feet?
7. A water balloon is dropped from a height of 26 feet. How long before it lands on someone who is 6 feet tall?
8. A potato is launched from the ground with an initial velocity of 15 m/sec. What is its maximum height?

Solving Quadratic Inequalities in One Variable

Example:

Solve $x^2-2x-3>0$.

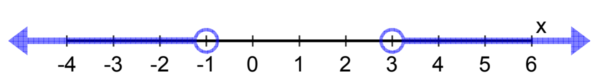
Find where the expression on the left side of the inequality equals zero.

$$x^2-2x-3=0$$
$$(x-3)(x+1)=0$$
$$x-3=0 \text{ or } x+1=0$$
$$x=3 \text{ or } x=-1$$

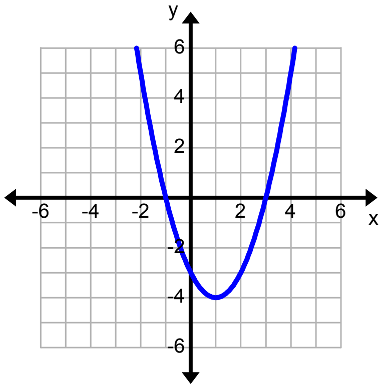
We are asked to find where the expression is greater than zero, in other words, where the expression is positive. Determine if the expression is positive or negative around each zero. Select a value in the interval and evaluate the expression at that value, then decide if the result is positive or negative.

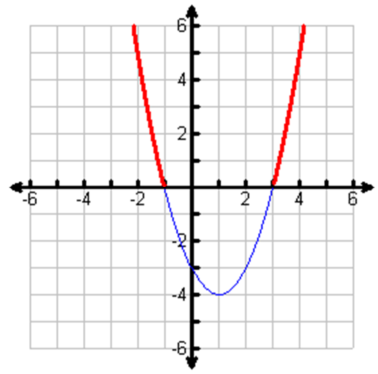
$x<-1$	$-1<x<3$	$x>3$
$x=-2$ $(-2)^2-2(-2)-3$ $4-(-4)-3$ $4+4-3$ 5 Positive	$x=0$ $(0)^2-2(0)-3$ $0-0-3$ $0-3$ -3 Negative	$x=4$ $(4)^2-2(4)-3$ $16-8-3$ $8-3$ 5 Positive

There are two intervals where the expression is positive: when $x<-1$ and when $x>3$. Therefore, the answer to the inequality is $(-\infty,-1)\cup(3,\infty)$. The answer could be represented on a number line as follows:



Look at the graph of the function $f(x)=x^2-2x-3$. Determine the intervals where the function is positive.





The function is positive $(-\infty,-1)\cup(3,\infty)$. Notice that this interval is the same interval we obtained when we tested values around the zeros of the expression.

Example:

Solve $-3x^2+5x+2\le 0$.

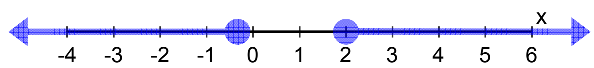
Find where the expression on the left side of the inequality equals zero.

$$-3x^2+5x+2=0$$
$$-\left(3x^2-5x-2\right)=0$$
$$-(3x+1)(x-2)=0$$
$$3x+1=0 \text{ or } x-2=0$$
$$3x=-1 \text{ or } x=2$$
$$x=-\frac{1}{3} \text{ or } x=2$$

We are asked to find where the expression is less than or equal to zero, in other words, where the expression is negative or zero. Determine if the expression is positive or negative around each zero. Select a value in the interval and evaluate the expression at that value, then decide if the result is positive or negative.

$x<-\frac{1}{3}$	$-\frac{1}{3}<x<2$	$x>2$
$x=-1$ $-3(-1)^2+5(-1)+2$ $-3\cdot 1+(-5)+2$ $-3-5+2$ -6 Negative	$x=0$ $-3(0)^2+5(0)+2$ $-3\cdot 0+0+2$ $0+0+2$ 2 Positive	$x=3$ $-3(3)^2+5(3)+2$ $-3\cdot 9+15+2$ $-27+15+2$ -10 Negative

There are two intervals where the expression is negative: when $x < -\frac{1}{3}$ and when $x > 2$. It equals zero at $x = -\frac{1}{3}$ and $x = 2$. Therefore, the answer to the inequality is $\left(-\infty, -\frac{1}{3}\right] \cup \left[-2, \infty\right)$. The answer could be represented on a number line as follows:



Example:

Solve $\dfrac{2}{x+4} < 0$.

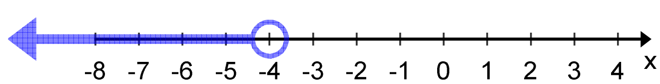
Find where the denominator is equal to zero.

$$x + 4 = 0$$
$$x = -4$$

We are asked to find where the expression is less than zero, in other words, where the expression is negative. Determine if the expression is positive or negative around where the denominator is equal to zero. Select a value in the interval and evaluate the expression at that value, then decide if the result is positive or negative.

$x < -4$	$x > -4$
$x = -5$ $\dfrac{2}{-5+4}$ $\dfrac{2}{-1}$ -2 Negative	$x = -3$ $\dfrac{2}{-3+4}$ $\dfrac{2}{1}$ 2 Positive

The function is negative when $x < -4$. Therefore, the answer to the inequality is $(-\infty, -4)$. The answer could be represented on a number line as follows:



Example:

Solve $\dfrac{x+3}{2x-5} \geq 0$.

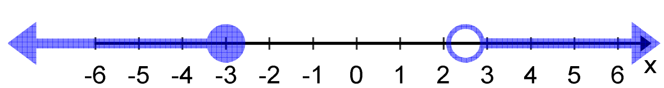
Find where both the numerator and the denominator are equal to zero.

$$x+3=0 \qquad\qquad 2x-5=0$$
$$x=-3 \qquad\qquad x=\frac{5}{2}$$

We are asked to find where the expression is greater than or equal to zero, in other words, where the expression is positive. Determine if the expression is positive or negative around where the numerator and denominator are equal to zero. Select a value in the interval and evaluate the expression at that value, then decide if the result is positive or negative.

$x<-3$	$-3<x<\frac{5}{2}$	$x>\frac{5}{2}$
$x=-4$ $\frac{x+3}{2x-5}$ $\frac{-4+3}{2(-4)-5}$ $\frac{-1}{-13}$ $\frac{1}{13}$ Positive	$x=0$ $\frac{x+3}{2x-5}$ $\frac{0+3}{2(0)-5}$ $\frac{3}{-5}$ $-\frac{3}{5}$ Negative	$x=3$ $\frac{x+3}{2x-5}$ $\frac{3+3}{2(3)-5}$ $\frac{6}{1}$ 6 Positive

There are two intervals where the expression is positive: when $x<-3$ and when $x>\frac{5}{2}$. The function is only equal to zero when $x=-3$ because the denominator cannot equal zero. Therefore, the answer to the inequality is $(-\infty,-3]\cup\left(\frac{5}{2},\infty\right)$. The answer could be represented on a number line as follows:



Example:

A rocket is launched with an initial velocity of 160 ft/sec from a 4 foot high platform. How long is the rocket at least 260 feet?

Write an inequality to represent the situation. The initial velocity, v_0, is 160 ft/sec and the initial height, h_0, is 4 feet. The height will be at least (greater than or equal to) 260 ft.

$$-16t^2 + v_0 t + h_0 \geq h$$
$$-16t^2 + 160t + 4 \geq 260$$

You need to determine the real world domain for this situation. Time is the independent variable. It starts at 0 seconds and ends when the rocket hits the ground at 10 seconds.

Move all the terms to one side of the inequality so the expression is compared to zero.

$$-16t^2 + 160t - 256 \geq 0$$

Find where the expression is equal to zero.

$$-16t^2 + 160t - 256 = 0$$
$$-16\left(t^2 - 10t + 16\right) = 0$$
$$-16(t-2)(t-8) = 0$$
$$t - 2 = 0 \text{ or } t - 8 = 0$$
$$t = 2 \text{ or } t = 8$$

Determine if the expression is positive or negative around each zero. Select a value in the interval and evaluate the expression at that value, then decide if the result is positive or negative.

$0 < t < 2$	$2 < t < 8$	$8 < t < 10$
$t = 1$	$t = 3$	$t = 9$
$-16t^2 + 160t - 256$	$-16t^2 + 160t - 256$	$-16t^2 + 160t - 256$
$-16(1)^2 + 160(1) - 256$	$-16(3)^2 + 160(3) - 256$	$-16(9)^2 + 160(9) - 256$
$-16 \cdot 1 + 160(1) - 256$	$-16 \cdot 9 + 480 - 256$	$-16 \cdot 81 + 1440 - 256$
$-16 + 160 - 256$	$-144 + 480 - 256$	$-1296 + 1440 - 256$
-112	80	-112
negative	Positive	negative

The rocket is at or above 260 feet from $t = 2$ seconds to $t = 8$ seconds. The difference is 6, so the rocket is at least 260 feet for 6 seconds.

Practice Exercises C

Solve.

1. $x^2+18x+80>0$
2. $x^2-11x+30<0$
3. $x^2+6x\geq -8$
4. $x^2-4x\leq 3$
5. $3x^2+6x<0$
6. $3x^2+17x+10\geq 0$
7. $\dfrac{1}{x-5}<0$
8. $\dfrac{x-2}{x+4}>0$
9. $\dfrac{2x+1}{x-3}\leq 0$
10. A bottle of water is thrown upward with an initial velocity of 32 ft/sec from a cliff that is 1920 feet high. For what time does the height exceed 1920 feet?
11. A company determines that its total profit function can be modeled by $P(x)=-2x^2+480x-16{,}000$. Find all values of x for which it makes a profit.
12. A rocket is launched with an initial velocity of 24 m/sec from a platform that is 3 meters high. The rocket will burst into flames unless it stays below 25 meters. Find the interval of time before the rocket bursts into flames.

Solving for a Specified Variable

<table>
<tr><td colspan="2">Sometimes it is necessary to use algebraic rules to manipulate formulas in order to work with a variable imbedded within the formula.</td></tr>
<tr><td>Given the area of a circle, solve for the radius.

Given: $A = \pi r^2$</td><td>Solve for r: $\frac{A}{\pi} = r^2$
$\pm\sqrt{\frac{A}{\pi}} = r$</td></tr>
<tr><td>Surface area of a right cylindrical solid with radius r and height h

$A = 2\pi r^2 + 2\pi rh$</td><td>You may have to use the quadratic formula.
Solve for r:
$A = 2\pi r^2 + 2\pi rh$
$0 = 2\pi r^2 + 2\pi rh - A$
$a = 2\pi \quad b = 2\pi h \quad c = -A$
$r = \frac{-2\pi h \pm \sqrt{(2\pi h)^2 - 4(2\pi)(-A)}}{2(2\pi)}$
$r = \frac{-2\pi h \pm \sqrt{4\pi^2 h^2 + 8\pi A}}{4\pi}$
$r = \frac{-2\pi h \pm 2\sqrt{\pi^2 h^2 + 2\pi A}}{4\pi}$
$r = \frac{-\pi h \pm \sqrt{\pi^2 h^2 + 2\pi A}}{2\pi}$</td></tr>
</table>

Practice Exercises D
Solve for the indicated variable.

1. $a^2 + b^2 = c^2$; solve for b
2. $S = 2hl + 2hw + 2lw$; solve for h
3. $A = 6s^2$; solve for s
4. $A = A_0(1-r)^2$; solve for r
5. $N = \frac{k^2 - 3k}{2}$; solve for k
6. $F = \frac{Gm_1 m_2}{r^2}$; solve for r
7. $N = \frac{1}{2}(n^2 - n)$; solve for n
8. $(x+1)^2 + (y-3)^2 = r^2$; solve for y

Unit 3 Cluster 3 Honors: Polynomial and Rational Inequalities

Cluster 3: Creating equations that describe numbers or relationships
H.1.2 Solve polynomial and rational inequalities in one variable.

Example:

Solve $x^4 - 13x^2 + 30 < 0$.

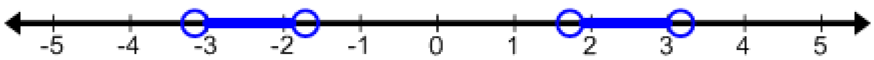
$x^4 - 13x^2 + 30 < 0$	Find where $x^4 - 13x^2 + 30 = 0$.
$x^4 - 13x^2 + 30 = 0$ $(x^2 - 10)(x^2 - 3) = 0$ $x^2 - 10 = 0$ $x^2 - 3 = 0$ $x^2 = 10$ or $x^2 = 3$ $x = \pm\sqrt{10}$ $x = \pm\sqrt{3}$	This expression is quadratic in nature. Factor the expression using that technique and use the zero product property to solve for each factor.

Test around each zero to determine if the expression is positive or negative on the interval.

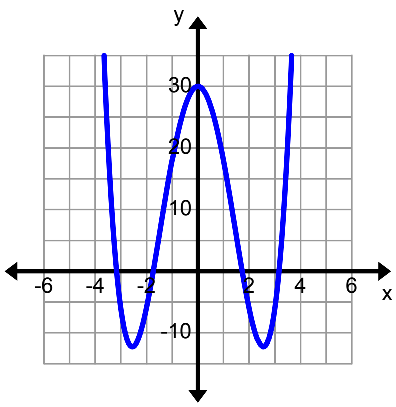
Interval	**Test Point**	**Expression evaluated at point**	**Positive/Negative**
$x < -\sqrt{10}$	$x = -4$	$(-4)^4 - 13(-4)^2 + 30$ 78	Positive
$-\sqrt{10} < x < -\sqrt{3}$	$x = -2$	$(-2)^4 - 13(-2)^2 + 30$ -6	Negative
$-\sqrt{3} < x < \sqrt{3}$	$x = 0$	$(0)^4 - 13(0)^2 + 30$ 30	Positive
$\sqrt{3} < x < \sqrt{10}$	$x = 2$	$2^4 - 13(2)^2 + 30$ -6	Negative
$x > \sqrt{10}$	$x = 4$	$(4)^4 - 13(4)^2 + 30$ 78	Positive

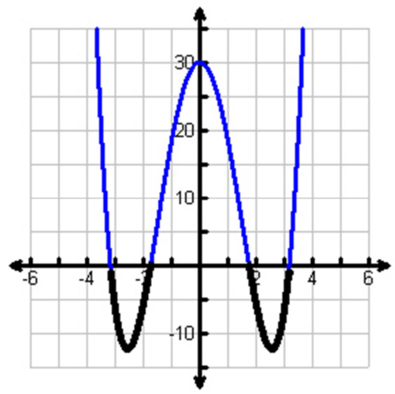
The expression is less than zero when it is negative.
The expression is negative on the intervals $-\sqrt{10} < x < -\sqrt{3}$ and $\sqrt{3} < x < \sqrt{10}$.
The answer can also be written as $\left(-\sqrt{10}, -\sqrt{3}\right) \cup \left(\sqrt{10}, \sqrt{3}\right)$.

The answer could also be represented on a number line.



Looking at the graph, the intervals that satisfy this inequality are the parts of the function below the *x*-axis. Notice the intervals are the same.





Example:

Solve $x^3+6x^2 \ge 0$

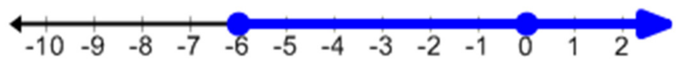
$x^3+6x^2 \ge 0$	Find where $x^3+6x^2=0$
$x^3+6x^2=0$ $x^2(x+6)=0$ $x^2=0$ or $x+6=0$ $x=0$ $\quad$ $x=-6$	Factor the expression and use the zero product property to solve for each factor.

Test around each zero to determine if the expression is positive or negative on the interval.

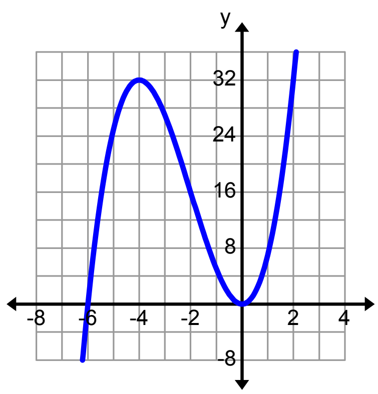
Interval	Test Point	Expression evaluated at point	Positive/Negative
$x<-6$	$x=-10$	$(-10)^3+6(-10)^2$ -400	Negative
$-6<x<0$	$x=-1$	$(-1)^3+6(-1)^2$ 5	Positive
$x>0$	$x=5$	$(5)^3+6(5)^2$ 275	Positive

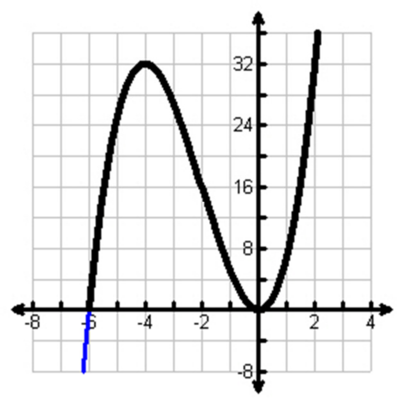
The expression is greater than zero when it is positive.
The expression is positive on the intervals $-6<x<0$ and $x<0$. Keep in mind it is also equal to zero so the endpoints are also included. The intervals that satisfy the inequality are $-6\le x\le 0$ and $x\le 0$. The interval would be written $x\ge -6$.
The answer can also be written as $[-6,\infty)$.

The answer could also be represented on a number line.



Looking at the graph, the intervals that satisfy this inequality are the parts of the function above the *x*-axis, including the values on the *x*-axis. Notice the intervals are the same.





Example:

$(x-1)(x+2)(x-4)\le 0$

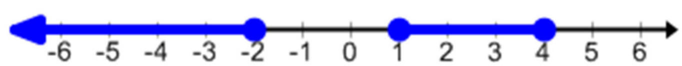
$(x-1)(x+2)(x-4)\le 0$	Find where $(x-1)(x+2)(x-4)=0$
$(x-1)(x+2)(x-4)=0$ $x-1=0$ or $x+2=0$ or $x-4=0$ $x=1$ $\quad x=-2$ $\quad x=4$	Use the zero product property to solve for each factor.

Test around each zero to determine if the expression is positive or negative on the interval.

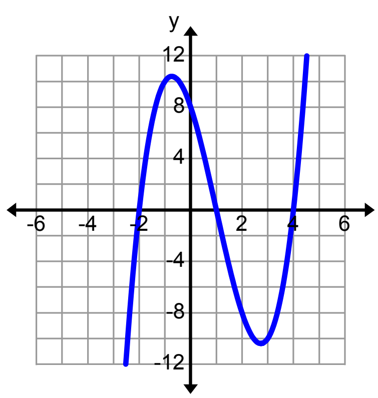
Interval	**Test Point**	**Expression evaluated at point**	**Positive/Negative**
$x < -2$	$x = -5$	$((-5)-1)((-5)+2)((-5)-4)$ -162	Negative
$-2 < x < 1$	$x = 0$	$(0-1)(0+2)(0-4)$ 8	Positive
$1 < x < 4$	$x = 2$	$(2-1)(2+2)(2-4)$ -8	Negative
$x > 4$	$x = 6$	$(6-1)(6+2)(6-4)$ 80	Positive

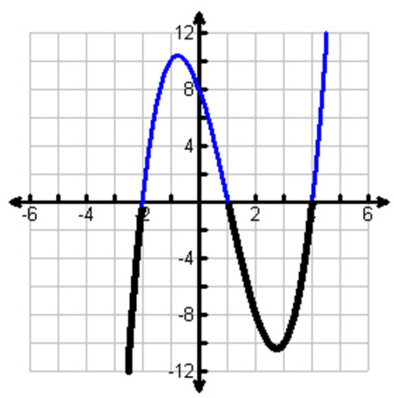
The expression is less than zero when it is negative.
The expression is negative on the intervals $x < -2$ and $1 < x < 4$. Keep in mind it is also equal to zero so the endpoints are also included. The intervals that satisfy the inequality are $x \le -2$ and $1 \le x \le 4$.
The answer can also be written $(-\infty, -2] \cup [1, 4]$.

The answer could also be represented on a number line.



Looking at the graph, the intervals that satisfy this inequality are the parts of the function below the *x*-axis, including the values on the *x*-axis. Notice the intervals are the same.





Example:

$$\frac{2x+5}{x+1} \geq \frac{x-1}{x+1}$$

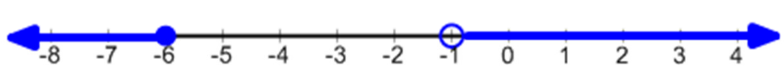
$\frac{2x+5}{x+1} \geq \frac{x-1}{x+1}$ $\frac{2x+5}{x+1} - \frac{x-1}{x+1} \geq 0$ $\frac{2x+5-(x-1)}{x+1} \geq 0$ $\frac{2x+5-x+1}{x+1} \geq 0$ $\frac{x+6}{x+1} \geq 0$	Compare the inequality to zero. Make sure the denominators (the bottoms) are the same. Combine the numerators (the tops).
$x+6=0$ or $x+1=0$ $x=-6$ $x=-1$	Set all of the factors in the numerator and denominator equal to zero, and then solve.

Test around each zero to determine if the expression is positive or negative on the interval.

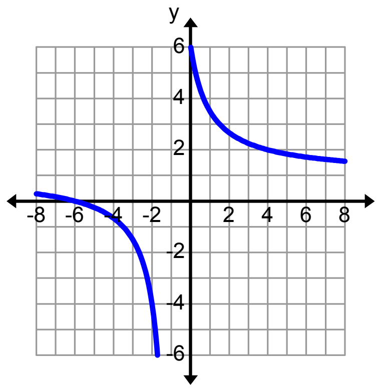
Interval	Test Point	Expression evaluated at point	Positive/Negative
$x<-6$	$x=-10$	$\frac{-10+6}{-10+1}=\frac{-4}{-9}$ $\frac{4}{9}$	Positive
$-6<x<-1$	$x=-3$	$\frac{-3+6}{-3+1}=\frac{3}{-2}$ $-\frac{3}{2}$	Negative
$x>-1$	$x=0$	$\frac{0+6}{0+1}=\frac{6}{1}$ 6	Positive

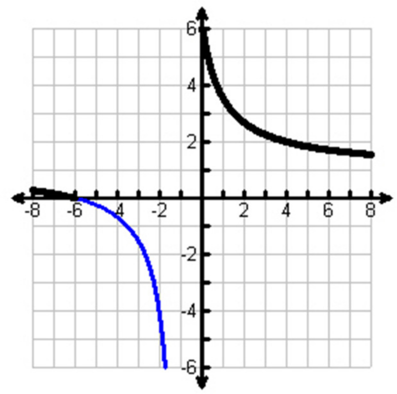
The expression is greater than zero when it is positive.
The expression is positive on the intervals $x<-6$ and $x>-1$. Keep in mind it is also equal to zero so the endpoints are also included except for the denominator which cannot be equal to zero. The intervals that satisfy the inequality are $x \leq -6$ and $x>-1$.
The answer can also be written as $(-\infty,-6] \cup (-1,\infty)$.

The answer could also be represented on a number line.



Looking at the graph, the intervals that satisfy this inequality are the parts of the function above the x-axis, including the values on the x-axis. Notice the intervals are the same.





Example:

$$\frac{x+3}{x-3} > -1$$

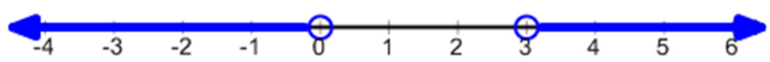
$\frac{x+3}{x-3} > -1$ $\frac{x+3}{x-3} + 1 > 0$ $\frac{x+3}{x-3} + 1 \cdot \left(\frac{x-3}{x-3}\right) > 0$ $\frac{x+3}{x-3} + \frac{x-3}{x-3} > 0$ $\frac{2x}{x-3} > 0$	Compare the inequality to zero. Make sure the denominators (the bottoms) are the same. Combine the numerators (the tops).
$2x = 0$ or $x - 3 = 0$ $x = 0$ $x = 3$	Set all of the factors in the numerator and denominator equal to zero, and then solve.

Test around each zero to determine if the expression is positive or negative on the interval.

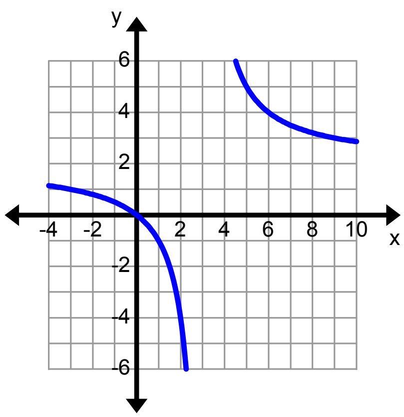
Interval	Test Point	Expression evaluated at point	Positive/Negative
$x<0$	$x=-5$	$\frac{2(-5)}{(-5)-3}=\frac{-10}{-8}$ $\frac{5}{4}$	Positive
$0<x<3$	$x=1$	$\frac{2(1)}{1-3}=\frac{2}{-2}$ -1	Negative
$x>3$	$x=5$	$\frac{2(5)}{(5)-3}=\frac{10}{2}$ 5	Positive

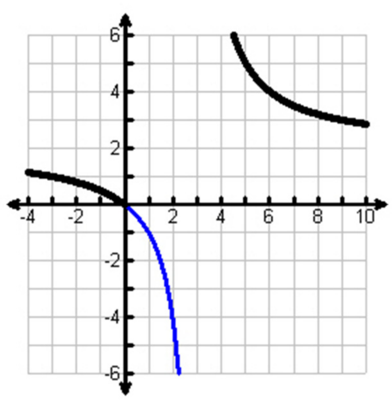
The expression is greater than zero when it is positive.
The expression is positive on the intervals $x<0$ and $x>3$. Keep in mind the denominator cannot be equal to zero.
The answer can also be written as $(-\infty,0)\cup(3,\infty)$.

The answer could also be represented on a number line.



Looking at the graph, the intervals that satisfy this inequality are the parts of the function above the x-axis, including the values on the x-axis. Notice the intervals are the same.





Example:

$$\frac{4}{x-2}<\frac{3}{x+1}$$

$\frac{4}{x-2}<\frac{3}{x+1}$ $\frac{4}{x-2}-\frac{3}{x+1}<0$ $\frac{4}{x-2}\cdot\left(\frac{x+1}{x+1}\right)-\frac{3}{x+1}\cdot\left(\frac{x-2}{x-2}\right)<0$ $\frac{4(x+1)}{(x-2)(x+1)}-\frac{3(x-2)}{(x-2)(x+1)}<0$ $\frac{4x+4-3x+6}{(x-2)(x+1)}<0$ $\frac{x+10}{(x-2)(x+1)}<0$	Compare the inequality to zero. Make sure the denominators (the bottoms) are the same. Combine the numerators (the tops).
$x+10=0$ or $x-2=0$ or $x+1=0$ $x=-10$ $x=2$ $x=-1$	Set all of the factors in the numerator and denominator equal to zero, and then solve.

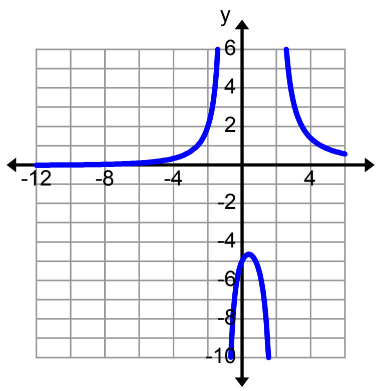
Test around each zero to determine if the expression is positive or negative on the interval.

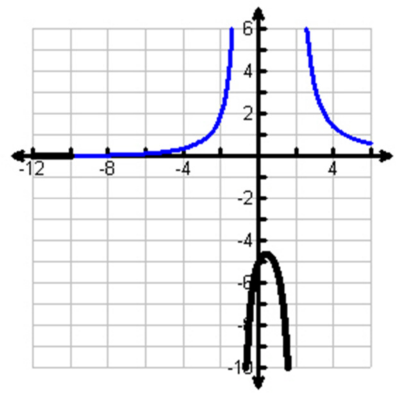
Interval	**Test Point**	**Expression evaluated at point**	**Positive/Negative**
$x<-10$	$x=-15$	$\frac{-15+10}{(-15-2)(-15+1)}$ $\frac{-5}{238}$	Negative
$-10<x<-1$	$x=-5$	$\frac{-5+10}{(-5-2)(-5+1)}$ $\frac{5}{28}$	Positive
$-1<x<2$	$x=0$	$\frac{0+10}{(0-2)(0+1)}=\frac{10}{-2}$ -5	Negative
$x>2$	$x=5$	$\frac{5+10}{(5-2)(5+1)}=\frac{15}{18}$ $\frac{5}{6}$	Positive

The expression is less than zero when it is negative.
The expression is negative on the intervals $x < -10$ and $-1 < x < 2$. Keep in mind the denominator cannot be equal to zero.
The answer can also be written as $(-\infty, -10) \cup (-1, 2)$.

The answer could also be represented on a number line.

Looking at the graph, the intervals that satisfy this inequality are the parts of the function below the *x*-axis. Notice the intervals are the same.





Practice Exercises A

Solve.

1. $(x-3)(x+4)(x+1) > 0$
2. $(x-5)(x-1)(x+2) \le 0$
3. $x^3 - 25x \le 0$
4. $x^4 - x^2 \ge 2$
5. $x^3 > x^2$
6. $x^4 < 4x^2$
7. $\dfrac{x}{x+2} \ge \dfrac{6}{x+2}$
8. $\dfrac{4x+5}{x+2} \ge 3$
9. $\dfrac{(x+3)(x-2)}{(x-1)^2} > 0$
10. $\dfrac{5x-7}{x-2} < \dfrac{8}{x-2}$
11. $\dfrac{1}{x-2} \ge \dfrac{1}{x+3}$
12. $\dfrac{5}{x+5} \le -\dfrac{3}{2x+1}$

You Decide

Carter's spaceship is trapped in a gravitational field of a newly discovered Class M planet. Carter will be in danger if his spaceship acceleration exceeds 500 m/h/h. If his acceleration can be modeled by the equation $A(t) = \dfrac{2500}{(5-t)^2}$ m/h/h . For what range of time is Carter's spaceship below the danger zone?

Unit 3 Cluster 3 (A.CED.2)
Writing and Graphing Equations in Two Variables

Cluster 3: Creating equations that describe numbers or relationships
3.3.2 Write and graph equations in 2 or more variables with labels and scales

Writing Quadratic Functions Given Key Features

A quadratic function can be expressed in several ways to highlight key features.

Vertex form: $f(x) = a(x-h)^2 + k$ highlights the vertex (h,k).

Factored from: $f(x) = a(x-p)(x-q)$ highlights the *x*-intercepts $(p,0)$ and $(q,0)$.

It is possible to write a quadratic function when given key features such as the vertex or the *x*-intercepts and another point on the graph of the parabola.

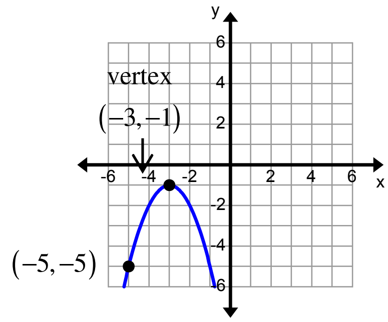
Example:

Write a quadratic equation for a parabola that has its vertex at $(2,4)$ and passes through the point $(1,6)$.

Answer:

$f(x) = a(x-h)^2 + k$	You are given the vertex which is a key feature that is highlighted by the vertex form of a quadratic function. Use this form to help you write the equation for the parabola graphed.
$f(x) = a(x-2)^2 + 4$	The vertex is $(2,4)$. $h = 2$ and $k = 4$. Substitute these values into the equation and simplify if necessary.
$6 = a(1-2)^2 + 4$	Use the point $(1,6)$ to help you find the value of *a*. The value of the function is 6 when $x = 1$ so substitute 1 in for *x* and 6 in for *f*(*x*).
$6 = a(1-2)^2 + 4$ $6 = a(-1)^2 + 4$ $6 = a \cdot 1 + 4$ $6 = a + 4$ $2 = a$	Use order of operations to simplify the expression on the right side of the equation then solve for *a*.
$f(x) = 2(x-2)^2 + 4$	Rewrite the expression substituting in the value for *a*.

Example: Write an equation for the parabola graphed below.



Answer:

$f(x) = a(x-h)^2 + k$	You are given the vertex which is a key feature that is highlighted by the vertex form of a quadratic function. Use this form to help you write the equation for the parabola graphed.
$f(x) = a(x-(-3))^2 + (-1)$ $f(x) = a(x+3)^2 - 1$	The vertex is $(-3,-1)$. $h=-3$ and $k=-1$. Substitute these values into the equation and simplify if necessary.
$-5 = a(-5+3)^2 - 1$	Use the point $(-5,-5)$ to help you find the value of a. The value of the function is -5 when x = -5 so substitute -5 in for x and -5 in for $f(x)$.
$-5 = a(-2)^2 - 1$ $-5 = a \cdot 4 - 1$ $-5 = 4a - 1$ $-4 = 4a$ $-1 = a$	Use order of operations to simplify the expression on the right side of the equation then solve for a.
$f(x) = -(x+3)^2 - 1$	Rewrite the expression substituting in the value for a.

Example:

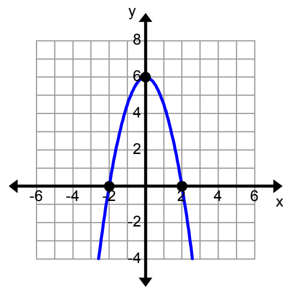
Write an equation for a parabola with *x*-intercepts $(-3,0)$ and $(5,0)$ and passes through the point $(3,-3)$.

Answer:

$f(x)=a(x-p)(x-q)$	You are given the *x*-intercepts which are a key feature that is highlighted by the factored form of a quadratic function. Use this form to help you write the equation for the parabola graphed.
$f(x)=a(x-(-3))(x-5)$ $f(x)=a(x+3)(x-5)$	One *x*-intercept is $(-3,0)$ so $p=-3$. The other *x*-intercept is $(5,0)$ $q=5$. Substitute these values into the equation and simplify if necessary.
$-3=a(3+3)(3-5)$	Use the point $(3,-3)$ to help you find the value of *a*. The value of the function is -3 when *x* = 3 so substitute 3 in for each *x* and -3 in for *f*(*x*).
$-3=a(3+3)(3-5)$ $-3=a(6)(-2)$ $-3=-12a$ $\frac{-3}{-12}=a$ $\frac{1}{4}=a$	Use order of operations to simplify the expression on the right side of the equation then solve for *a*.
$f(x)=\frac{1}{4}(x+3)(x-5)$	Rewrite the expression substituting in the value for *a*.

Example:

Write an equation for the parabola graphed below.



Answer:

$f(x) = a(x-p)(x-q)$	You are given the *x*-intercepts which are a key feature that is highlighted by the factored form of a quadratic function. Use this form to help you write the equation for the parabola graphed.
$f(x) = a(x-(-2))(x-2)$ $f(x) = a(x+2)(x-2)$	One *x*-intercept is $(-2,0)$ so $p=-2$. The other *x*-intercept is $(2,0)$ $q=2$. Substitute these values into the equation and simplify if necessary.
$6 = a(0+2)(0-2)$	Use the point $(0,6)$ to help you find the value of a. The value of the function is 6 when $x = 0$ so substitute 0 in for each x and 6 in for $f(x)$.
$6 = a(0+2)(0-2)$ $6 = a(2)(-2)$ $6 = -4a$ $\frac{6}{-4} = a$ $-\frac{3}{2} = a$	Use order of operations to simplify the expression on the right side of the equation then solve for a.
$f(x) = -\frac{3}{2}(x+2)(x-2)$	Rewrite the expression substituting in the value for a.

Practice Exercises A

Write a quadratic equation for the parabola described.

1. Vertex: $(2,3)$
   Point: $(0,7)$

2. Vertex: $(-1,4)$
   Point: $(1,8)$

3. Vertex: $(-3,-1)$
   Point: $(-2,0)$

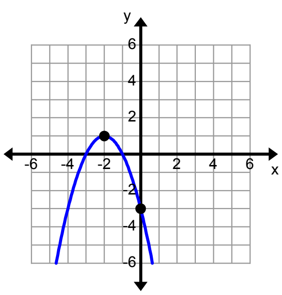
4. Intercepts: $(2,0)$ $(4,0)$
   Point: $(1,3)$

5. Intercepts: $(-1,0)$ $(7,0)$
   Point: $(5,-12)$

6. Intercepts: $(-5,0)$ $(4,0)$
   Point: $(3,8)$

Write a quadratic equation for the parabola graphed.

7.



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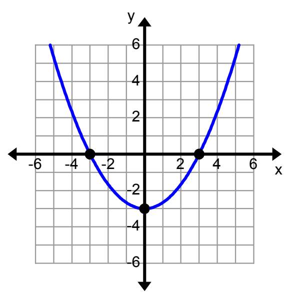

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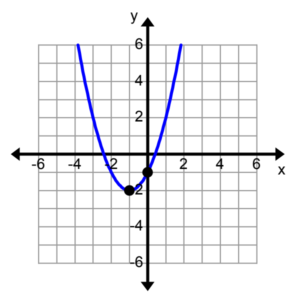

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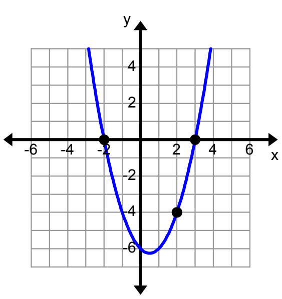

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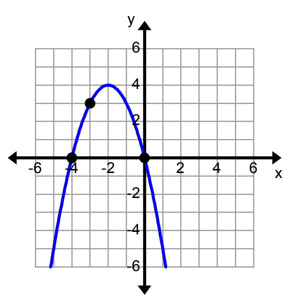

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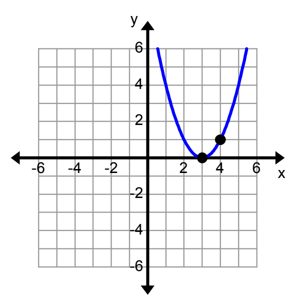
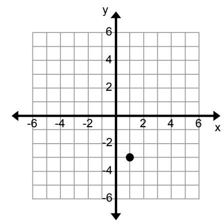
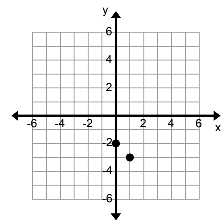
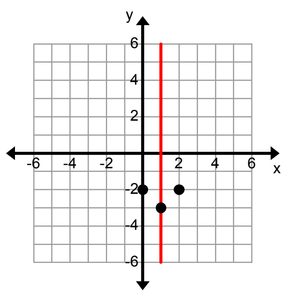
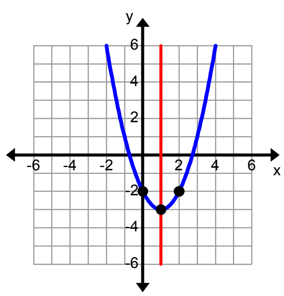
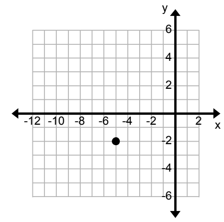
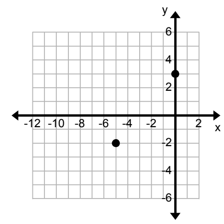
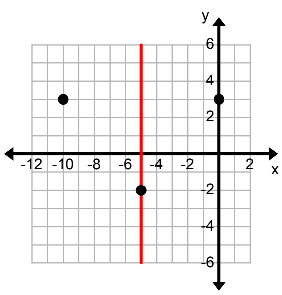
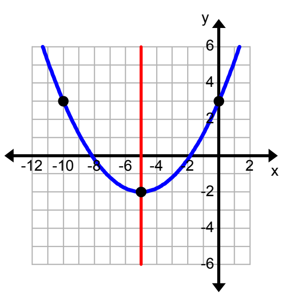
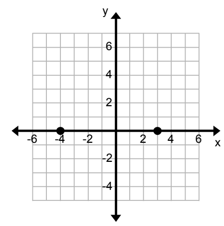
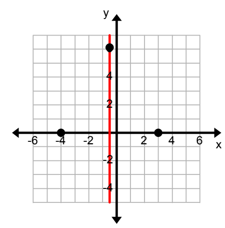
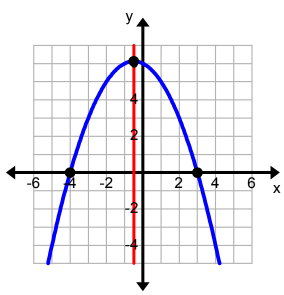


Graphing Quadratic Equations

Graphing from Standard Form $f(x) = ax^2 + bx + c$	
Example: $f(x) = x^2 - 2x - 2$	
$x = -\dfrac{b}{2a}$ $x = -\dfrac{(-2)}{2(1)} = 1$ $f(x) = x^2 - 2x - 2$ $f(1) = (1)^2 - 2(1) - 2$ $f(x) = -3$	Find the vertex. The vertex is (1, -3) Plot the vertex. 
$f(0) = (0)^2 - 2(0) - 2$ $f(0) = -2$	Find the *y*-intercept. The *y*-intercept is (0, -2) Plot the *y*-intercept. 
	Use the axis of symmetry to find another point that is the reflection of the *y*-intercept.
	Connect the points, drawing a smooth curve. Remember quadratic functions are "U" shaped.

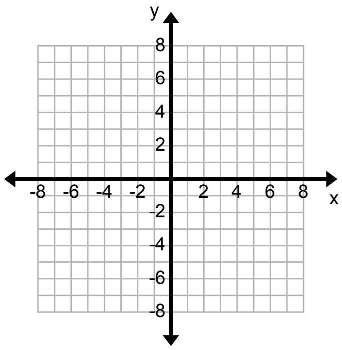
Graphing from Vertex Form $f(x) = a(x-h)^2 + k$	
Example: $f(x) = \frac{1}{5}(x+5)^2 - 2$	
$f(x) = a(x-h)^2 + k$ $f(x) = \frac{1}{5}(x+5)^2 - 2$ The vertex is (h, k).	Find the vertex. The vertex is (-5, -2) Plot the vertex. 
$f(0) = \frac{1}{5}(0+5)^2 - 2$ $f(0) = 3$	Find the y-intercept. The y-intercept is (0, 3) Plot the y-intercept. 
	Use the axis of symmetry to find another point that is the reflection of the y-intercept.
	Connect the points, drawing a smooth curve. Remember quadratic functions are “U” shaped.

Graphing from Factored Form $f(x)=a(x-p)(x-q)$	
Example: $$f(x)=-\frac{1}{2}(x+4)(x-3)$$	
$f(x)=-\frac{1}{2}(x+4)(x-3)$ $x+4=0 \qquad x-3=0$ $x=-4 \qquad x=3$	Find the x-intercepts. The x-intercepts are (-4, 0) and (3, 0) Plot the x-intercepts. 
$\dfrac{-4+3}{2}=-\dfrac{1}{2}$	Find the x-coordinate between the two intercepts.
$f\left(-\frac{1}{2}\right)=-\frac{1}{2}\left(-\frac{1}{2}+4\right)\left(-\frac{1}{2}-3\right)$ $f\left(-\frac{1}{2}\right)=-\frac{1}{2}\left(\frac{7}{2}\right)\left(-\frac{7}{2}\right)$ $f\left(-\frac{1}{2}\right)=\frac{49}{8}=6.125$	Use the x-coordinate to find the y-coordinate. The vertex is $\left(-\frac{1}{2},\frac{49}{8}\right)$ Plot the vertex. 
	Connect the points, drawing a smooth curve. Remember quadratic functions are "U" shaped.

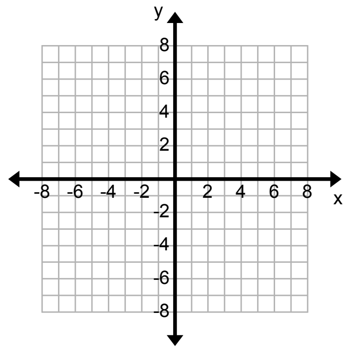
Practice Exercises B

Graph the following equations.

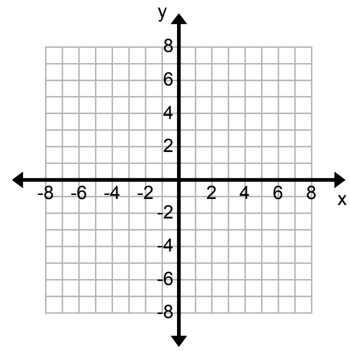
1. $f(x) = -x^2 + 6x - 6$



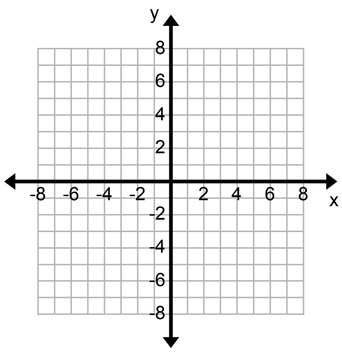
2. $f(x) = 2x^2 + 4x - 1$



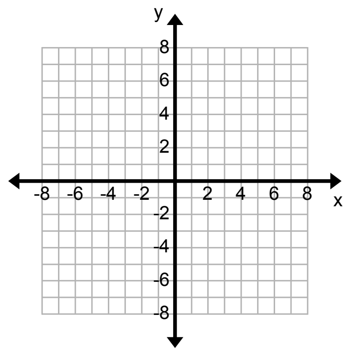
3. $f(x) = \frac{1}{3}x^2 + x + 2$



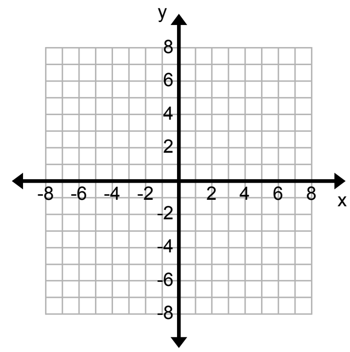
4. $f(x) = \frac{1}{2}(x-1)^2 - 3$



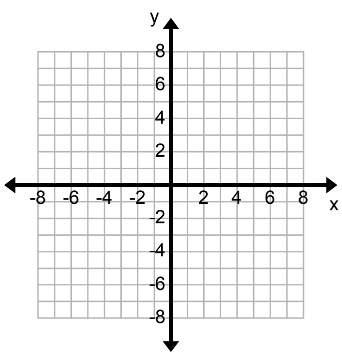
5. $f(x) = -(x+2)^2 + 5$



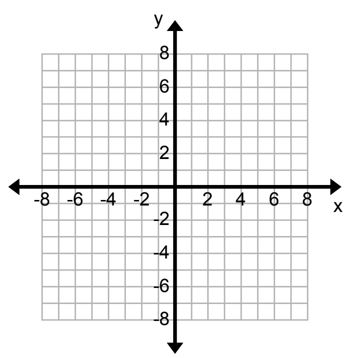
6. $f(x) = (x-3)^2 - 8$



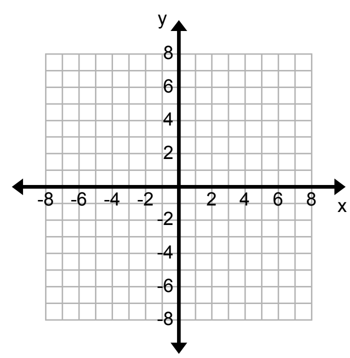
7. $f(x) = \frac{1}{2}(x+2)(x-5)$



8. $f(x) = (x+1)(x+5)$



9. $f(x) = -2(x-1)(x-3)$

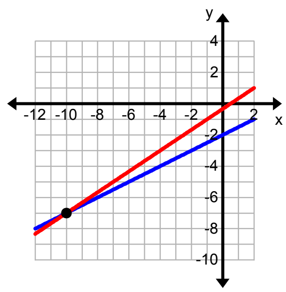


Unit 3 Cluster 6 (A.REI.7): Solve Systems of Equations

3.6.1 Solve simple systems containing linear and quadratic functions algebraically and graphically.

Recall solving systems of equations in Secondary 1. We are looking for the intersection of the two lines. There were three methods used. Below are examples of each method.

Solve: $$\begin{aligned} x-2y&=4 \\ 2x-3y&=1 \end{aligned}$$

Graphing	
	Graph the two equations and find the intersection. The intersection is (-10, -7).

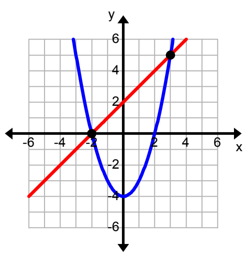
Substitution	
$x=2y+4$	1. Solve for x in the first equation.
$2(2y+4)-3y=1$	2. Substitute the solution for x in the second equation
$4y+8-3y=1$ $y+8=1$ $y=-7$	3. Solve for y
$x=2y+4$ $x=2(-7)+4$ $x=-14+4$ $x=-10$	4. Substitute y back into the first equation to solve for x.
The solution is (-10, -7)	5. The solution is the intersection.

Elimination	
$x-2y=4$ $2x-3y=1$ $-2(x-2y=4)$ $2x-3y=1$	1. In order to eliminate the x's, multiply the top equation by -2.
$-2x+4y=-8$ $2x-3y=1$ $y=-7$	2. Combine the two equations.
$x-2y=4$ $x-2(-7)=4$ $x+14=4$ $x=-10$	3. Substitute $y=-7$ into either original equation in order to solve for x
The intersection is (-10, -7)	4. The solution is the intersection.

We will use these methods to help solve systems involving quadratic equations.

Example:

Find the intersection of the following two equations:	$y=x^2-4$ $x-y=-2$

Graphing	
	Graph the two equations and find the intersection(s). The intersections are (-2, 0) and (3, 5).

Substitution	
$x-(x^2-4)=-2$	1. The first equation is already solved for y; substitute x^2-4 for y in the second equation.

$x+4=-2$ $-x^2+x+4=-2$ $-x^2+x+6=0$ $x^2-x-6=0$	2. Simplify and write in standard polynomial form.
$x^2-x-6=0$ $(x-3)(x+2)=0$ $x=3$ or $x=-2$	3. Solve for x using the method of your choice
$y=x^2-4$ $\quad$ $y=x^2-4$ $y=(3)^2-4$ $\quad$ $y=(-2)^2-4$ $y=9-4$ $\quad$ $y=4-4$ $y=5$ $\quad$ $y=0$	4. Substitute the x values back into the first equation to solve for y.
The solutions are (3, 5) and (-2, 0)	5. The solutions are the intersections.

Elimination $x^2+0x-4=y$ $x+2=y$	1. Line up like variables.
$x^2+0x-4=y$ $-1(x+2=y)$ $x^2+0x-4=y$ $-x-2=-y$ $x^2-x-6=0$	2. Multiply the second equation by -1 then combine the two equations.
$x^2-x-6=0$ $(x-3)(x+2)=0$ $x=3$ or $x=-2$	3. Solve using the method of your choice.
$y=x^2-4$ $\quad$ $y=x^2-4$ $y=(3)^2-4$ $\quad$ $y=(-2)^2-4$ $y=9-4$ $\quad$ $y=4-4$ $y=5$ $\quad$ $y=0$	4. Substitute the x values back into the first equation to solve for y.
The solutions are (3, 5) and (-2, 0)	5. The solutions are the intersections.

Example:

Using the method of your choice, find the intersection between the following equations: $$x^2 + y^2 = 4$$ $$2x + y = -1$$

Substitution $$y = -2x - 1$$	1. Solve for x in the second equation.
$$x^2 + (-2x-1)^2 = 4$$	2. Substitute $-2x-1$ for y in the first equation.
$$x^2 + 4x^2 + 4x + 1 = 4$$ $$5x^2 + 4x - 3 = 0$$ $$x = \frac{-4 \pm \sqrt{4^2 - 4(5)(-3)}}{2(5)}$$ $$x = \frac{-4 \pm \sqrt{16+60}}{10}$$ $$x = \frac{-4 \pm \sqrt{76}}{10}$$ $$x = \frac{-4 \pm 2\sqrt{19}}{10}$$ $$x = \frac{-2 \pm \sqrt{19}}{5}$$ $$x = \frac{-2+\sqrt{19}}{5} \text{ or } x = \frac{-2-\sqrt{19}}{5}$$ $$x \approx 0.472 \quad \text{or} \quad x \approx -1.272$$	3. Simplify and solve for x.
$$y = -2x - 1 \qquad y = -2x - 1$$ $$y = -2\left(\frac{-2+\sqrt{19}}{5}\right) - 1 \qquad y = -2\left(\frac{-2-\sqrt{19}}{5}\right) - 1$$ $$y = \frac{-1-2\sqrt{19}}{5} \qquad y = \frac{-1+2\sqrt{19}}{5}$$ $$y \approx -1.944 \qquad y \approx 1.544$$	4. Substitute the x values back into the first equation to solve for y.

The solutions are $\left(\frac{-2+\sqrt{19}}{5},\frac{-1-2\sqrt{19}}{5}\right)$ and $\left(\frac{-2-\sqrt{19}}{5},\frac{-1+2\sqrt{19}}{5}\right)$ Or approximately (0.472, -1.944) or (-1.272, 1.544)	5. The solutions are the intersections.

Practice Exercises A
Solve each of the systems of equations.

1. $\begin{array}{l} x+y=6 \\ y^2=x+3 \end{array}$

2. $\begin{array}{l} x^2+y^2=41 \\ y-x=1 \end{array}$

3. $\begin{array}{l} 3x+y=7 \\ 4x^2+5y=24 \end{array}$

4. $\begin{array}{l} y=x^2 \\ 3x=y+2 \end{array}$

5. $\begin{array}{l} y^2=x+3 \\ 2y=x+4 \end{array}$

6. $\begin{array}{l} 4x^2+9y^2=36 \\ 3y+2x=6 \end{array}$

7. $\begin{array}{l} y=x^2 \\ x=y^2 \end{array}$

8. $\begin{array}{l} 2x+y=1 \\ y=4-x^2 \end{array}$

9. $\begin{array}{l} x^2+y^2=89 \\ x-y=3 \end{array}$

10. $\begin{array}{l} x^2+y^2=45 \\ y-x=3 \end{array}$

11. $\begin{array}{l} x^2+y^2=14 \\ x^2-y^2=4 \end{array}$

12. $\begin{array}{l} (x-1)^2+(y+3)^2=4 \\ y=x \end{array}$

Unit 3 Cluster 6 Honors (A.REI.8 and A.REI.9)
Solving Systems of Equations with Vectors and Matrices

H.3.1 Represent a system of linear equations as a single matrix equation in a vector variable.
H.3.2 Find the inverse of a matrix if it exists and use it to solve systems of linear equations (using technology for matrices of dimension 3×3 or greater).

In Secondary Mathematics 1 Honors you learned to solve a system of two equations by writing the corresponding augmented matrix and using row operations to simplify the matrix so that it had ones down the diagonal from upper left to lower right, and zeros above and below the ones.

$$\left[\begin{array}{cc|c} 1 & 0 & g \\ 0 & 1 & h \end{array}\right]$$

When a matrix is in this form it is said to be in **reduced row-echelon form**. The process for simplifying a matrix to reduced row-echelon form is called **Gauss-Jordan elimination** after the two mathematicians, Carl Friedrich Gauss and Wilhelm Jordan. This process, using row operations, can be used for systems of two or more variables.

Row operations are listed below using the original matrix $\left[\begin{array}{ccc|c} 0 & -2 & -6 & 3 \\ -1 & -4 & 7 & 10 \\ 2 & 6 & 9 & -1 \end{array}\right]$.

Note: R indicates row and the number following indicates which row. For instance, $R1$ indicated row one.

Interchange Rows		
We want to have ones along the diagonal. We can switch rows so that the one ends up in the correct position.	Symbol: $R1 \leftrightarrow R2$ All of the elements of Row 1 will switch positions with all the elements of Row 2.	$\left[\begin{array}{ccc\|c} -1 & -4 & 7 & 10 \\ 0 & -2 & -6 & 3 \\ 2 & 6 & 9 & -1 \end{array}\right]$
Multiply a row by a scalar		
We want positive ones down the diagonal. Multiplying each element by a scalar k, $(k \neq 0)$, allows us to change values in one row while preserving the overall equality.	Symbol: $k \cdot R1$	$\left(-\frac{1}{2}\right)R1 = \left[\begin{array}{cccc} 0 & 1 & 3 & -\frac{3}{2} \end{array}\right]$ The matrix is now: $\left[\begin{array}{ccc\|c} 0 & 1 & 3 & -\frac{3}{2} \\ -1 & -4 & 7 & 10 \\ 2 & 6 & 9 & -1 \end{array}\right]$

Combine Two Rows		
You can add or subtract two rows to replace a single row. This enables you to get ones along the diagonal and zeros elsewhere.	$R2:R1+R2$	$R2:\begin{bmatrix}0 & -2 & -6 & 3\end{bmatrix}+\begin{bmatrix}-1 & -4 & 7 & 10\end{bmatrix}$ $R2:\begin{bmatrix}-1 & -6 & 1 & 13\end{bmatrix}$ The matrix is now: $\left[\begin{array}{ccc\|c}0 & -2 & -6 & 3\\ -1 & -6 & 1 & 13\\ 2 & 6 & 9 & -1\end{array}\right]$
Multiply by a scalar, then combine rows		
Sometimes it is necessary to multiply a row by a scalar before combining with another row to get a one or a zero where needed.	$R3:2\cdot R2+R3$	$R3:2\cdot\begin{bmatrix}-1 & -4 & 7 & 10\end{bmatrix}+\begin{bmatrix}2 & 6 & 9 & -1\end{bmatrix}$ $R3:\begin{bmatrix}-2 & -8 & 14 & 20\end{bmatrix}+\begin{bmatrix}2 & 6 & 9 & -1\end{bmatrix}$ $R3:\begin{bmatrix}0 & -2 & 23 & 19\end{bmatrix}$ The matrix is now: $\left[\begin{array}{ccc\|c}0 & -2 & -6 & 3\\ -1 & -4 & 7 & 10\\ 0 & -2 & 23 & 19\end{array}\right]$

Practice Exercises A
Perform the row operations on the given matrix.

$$\left[\begin{array}{ccc|c}3 & -3 & -1 & 6\\ 8 & -5 & 5 & -1\\ 0 & -6 & -9 & -3\end{array}\right]$$

1. $R2 \leftrightarrow R3$
2. $\left(\frac{1}{3}\right)\cdot R1$
3. $R2:(-1)\cdot R3+R2$

Using the matrix below, write appropriate row operation(s) to get the desired results. (Recall that a_{mn} indicates the element of the matrix is located in row *m* and column *n*.)

$$\left[\begin{array}{ccc|c}3 & -8 & 3 & -10\\ 2 & -5 & -10 & 9\\ 7 & -5 & 5 & -5\end{array}\right]$$

4. $a_{11}=1$
5. $a_{21}=0$
6. $a_{32}=0$

Example: Solve the following systems of equations using the Gauss-Jordan elimination method.

$$\begin{aligned} x-2y+z &= 7 \\ 3x-5y+z &= 14 \\ 2x-2y-z &= 3 \end{aligned}$$

$\left[\begin{array}{ccc\|c} 1 & -2 & 1 & 7 \\ 3 & -5 & 1 & 14 \\ 2 & -2 & -1 & 3 \end{array}\right]$	Rewrite the equations in matrix form.
$\left[\begin{array}{ccc\|c} 1 & -2 & 1 & 7 \\ 0 & 1 & -2 & -7 \\ 2 & -2 & -1 & 3 \end{array}\right]$	$R2: -3R1+R2$ $R2: [-3\ \ 6\ \ -3\ \ -21]+[3\ \ -5\ \ 1\ \ 14]$ $R2: [0\ \ 1\ \ -2\ \ -7]$
$\left[\begin{array}{ccc\|c} 1 & -2 & 1 & 7 \\ 0 & 1 & -2 & -7 \\ 0 & 2 & -3 & -11 \end{array}\right]$	$R3: -2R1+R3$ $R3: [-2\ \ 4\ \ -2\ \ -14]+[2\ \ -2\ \ -1\ \ 3]$ $R3: [0\ \ 2\ \ -3\ \ -11]$
$\left[\begin{array}{ccc\|c} 1 & 0 & -3 & -7 \\ 0 & 1 & -2 & -7 \\ 0 & 2 & -3 & -11 \end{array}\right]$	$R1: 2R2+R1$ $R1: [0\ \ 2\ \ -4\ \ -14]+[1\ \ -2\ \ 1\ \ 7]$ $R1: [1\ \ 0\ \ -3\ \ -7]$
$\left[\begin{array}{ccc\|c} 1 & 0 & -3 & -7 \\ 0 & 1 & -2 & -7 \\ 0 & 0 & 1 & 3 \end{array}\right]$	$R3: -2R2+R2$ $R3: [0\ \ -2\ \ 4\ \ 14]+[0\ \ 2\ \ -3\ \ -11]$ $R3: [0\ \ 0\ \ 1\ \ 3]$
$\left[\begin{array}{ccc\|c} 1 & 0 & 0 & 2 \\ 0 & 1 & -2 & -7 \\ 0 & 0 & 1 & 3 \end{array}\right]$	$R1: 3R3+R1$ $R1: [0\ \ 0\ \ 3\ \ 9]+[1\ \ 0\ \ -3\ \ -7]$ $R1: [1\ \ 0\ \ 0\ \ 2]$
$\left[\begin{array}{ccc\|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 3 \end{array}\right]$	$R2: 2R3+R2$ $R2: [0\ \ 0\ \ 2\ \ 6]+[0\ \ 1\ \ -2\ \ -7]$ $R2: [0\ \ 1\ \ 0\ \ -1]$
The solution is $(2,-1,3)$	Rewriting the answer in equation form you end up with: $x=2$ $y=-1$ $z=3$

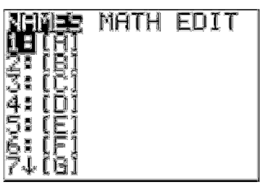
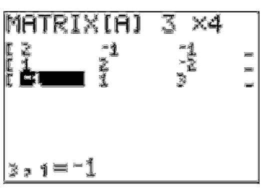
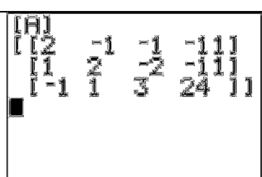
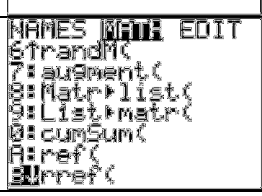
Practice Exercises B
Using Gauss-Jordan Elimination solve the following systems of equations.

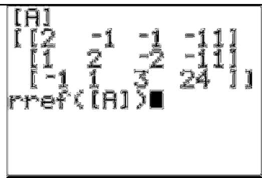
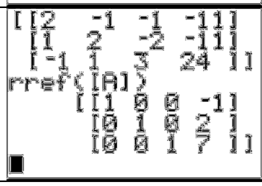
1. $$\begin{aligned} 3x - 2y &= 19 \\ 2x + y &= 1 \end{aligned}$$

2. $$\begin{aligned} 2x - y + 3z &= 13 \\ x + 2y + 2z &= 3 \\ 3x + y + z &= 4 \end{aligned}$$

Technology can be used to help solve systems of equations. The following system of equations can be solved using your graphing calculator.

$$\begin{aligned} 2x - y - z &= -11 \\ x + 2y - 2z &= -11 \\ -x + y + 3z &= 24 \end{aligned}$$

$\left[\begin{array}{ccc|c} 2 & -1 & -1 & -11 \\ 1 & 2 & -2 & -11 \\ -1 & 1 & 3 & 24 \end{array}\right]$	Write the augmented matrix.
	Enter the matrix into your calculator. Note: [A] is the default matrix Push $\boxed{\mathbf{2^{nd}}}$ $\boxed{x^{-1}}$ Arrow over to choose **EDIT** Push $\boxed{\textbf{ENTER}}$
	Type the dimensions. After each number push $\boxed{\textbf{ENTER}}$. For this example: $\boxed{\mathbf{3}}$ $\boxed{\textbf{ENTER}}$ $\boxed{\mathbf{4}}$ $\boxed{\textbf{ENTER}}$ Type each element in the first row. Push $\boxed{\textbf{ENTER}}$ after each number. Continue until every row has been entered.
	Push $\boxed{\mathbf{2^{nd}}}$ $\boxed{\textbf{MODE}}$ to return to the home screen Push $\boxed{\mathbf{2^{nd}}}$ $\boxed{x^{-1}}$ Then $\boxed{\textbf{ENTER}}$ $\boxed{\textbf{ENTER}}$ This will give you a chance to check your matrix for accuracy.
	Push $\boxed{\mathbf{2^{nd}}}$ $\boxed{x^{-1}}$ arrow over to MATH. Either arrow down to option **B** or push $\boxed{\text{ALPHA}}$ $\boxed{\text{APPS}}$ to get to **rref**(This is row reduced echelon form.

	Push $\boxed{\mathbf{2^{nd}}}$ $\boxed{x^{-1}}$ $\boxed{\textbf{ENTER}}$ $\boxed{)}$ Or select the matrix in which your equations are stored.
	Push $\boxed{\textbf{ENTER}}$ to obtain the answer.
The answer is (-1, 2, 7)	Rewriting the answer in equation form you end up with: $x=-1$ $y=2$ $z=7$

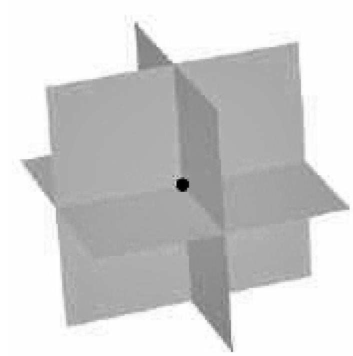
Practice Exercises C
Solve each of the following systems using technology.

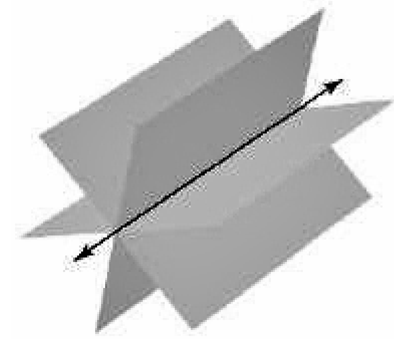
1. $\begin{aligned} 3x+y+2z&=31 \\ x+y+2z&=19 \\ x+3y+2z&=25 \end{aligned}$

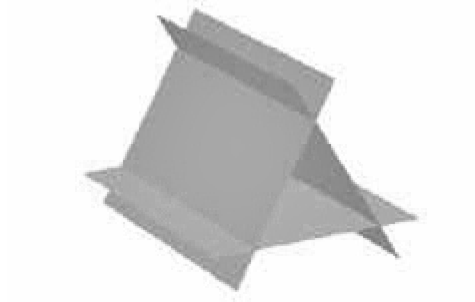
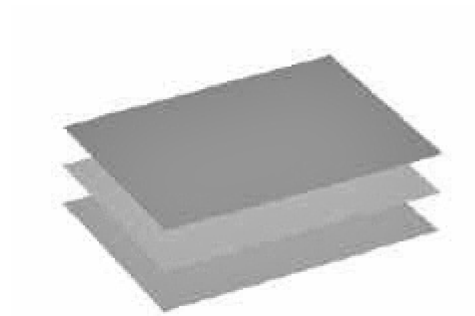
2. $\begin{aligned} 5x-2y-3z&=0 \\ x+y&=5 \\ 2x-3z&=4 \end{aligned}$

3. $\begin{aligned} 2x+y+2z&=2 \\ 3x-5y-z&=4 \\ x-2y-3z&=-6 \end{aligned}$

Not every system has a single point as the solution. The following situations may also occur.

Consistent and Independent $x-y+z=5$ $4x+2y+z=-1$ $9x+3y+z=13$	**A single point solution** $\left[\begin{array}{ccc\|c} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & -6 \\ 0 & 0 & 1 & -5 \end{array}\right]$ $(4,-6,-5)$	

Consistent and Dependent $$\begin{aligned} 6x - y - z &= 4 \\ -12x + 2y + 2z &= -8 \\ 5x + y - z &= 3 \end{aligned}$$	**A line solution defined by one variable** $$\left[\begin{array}{ccc\|c} 1 & 0 & -\frac{2}{11} & \frac{7}{11} \\ 0 & 1 & -\frac{1}{11} & -\frac{2}{11} \\ 0 & 0 & 0 & 0 \end{array}\right]$$ Rewriting the answer in equation form you end up with: $x - \frac{2}{11}z = \frac{7}{11}$ $y - \frac{1}{11}z = -\frac{2}{11}$ $0 = 0$ this is always true Note: z is an independent variable and can take on any real value The solution is written as: $\left(\frac{7}{11} + \frac{2}{11}z, -\frac{2}{11} + \frac{1}{11}z, z\right)$	
Consistent and Dependent $$\begin{aligned} x + y - 3z &= 1 \\ x - z - w &= 2 \\ 2x + y - 4z - w &= 3 \end{aligned}$$	**A plane solution defined by two variables** $$\left[\begin{array}{cccc\|c} 1 & 0 & -1 & -1 & 2 \\ 0 & 1 & -2 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$ Rewriting the answer in equation form you end up with: $x - z - w = 2$ $y - 2z + w = -1$ $0 = 0$ this is always true Note: z and w are independent variables and can take on any real value The solution is written as: $\left(2 + w + z, -1 - w + 2z, z, w\right)$	 All three planes coincide.

Inconsistent $x+y+z=6$ $2x-y-z=3$ $x+2y+2z=0$	**No solution** **There are no intersections common to all three planes or the three planes are parallel** $\left[\begin{array}{ccc\|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$ **The equations would be:** $x=0$ $y+z=0$ $0=1$ this is not true **No Solution**	 or 

Practice Exercises D
Solve the following systems. Indicate if the system is consistent or inconsistent.

1.
$$\begin{aligned} x-2y+z&=0 \\ 2x+2y-3z&=-3 \\ y-z&=-1 \\ -x+4y+2z&=13 \end{aligned}$$

2.
$$\begin{aligned} 5x+8y-6z&=14 \\ 3x+4y-2z&=8 \\ x+2y-2z&=3 \end{aligned}$$

3.
$$\begin{aligned} 5x+12y+z&=10 \\ 2x+5y+2z&=-1 \\ x+2y-3z&=5 \end{aligned}$$

4.
$$\begin{aligned} x-y-2z&=2 \\ 2x-3y+6z&=5 \\ 3x-4y+4z&=12 \end{aligned}$$

5.
$$\begin{aligned} x-y+2w&=3 \\ 3x-2y-w&=4 \\ x-3y+z+3w&=1 \\ 2x-4y-z+w&=-2 \end{aligned}$$

6.
$$\begin{aligned} 2x+3y-z+w&=7 \\ 2x-3y+z&=4 \\ 4x-y-w&=-3 \end{aligned}$$

Find the Inverse of a Matrix

Two $n \times n$ matrices are **inverses** of one another if their product is the $n \times n$ identity matrix. Not all matrices have an inverse. An $n \times n$ matrix has an inverse if and only if the determinant is **not** zero. The inverse of A is denoted by A^{-1}. There are two ways to find the inverse both can be done using technology.

$$A = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 3 & 4 \\ 0 & 4 & 3 \end{bmatrix} \qquad \begin{vmatrix} 1 & 1 & 0 \\ -1 & 3 & 4 \\ 0 & 4 & 3 \end{vmatrix} = -4$$

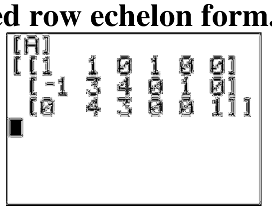
The determinant is not zero therefore the matrix has an inverse.

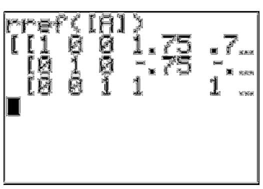
Method 1:

Rewrite the matrix with the 3×3 identity matrix.

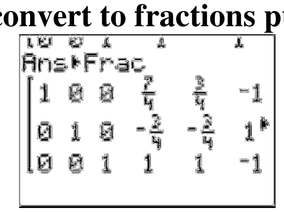
$$\left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ -1 & 3 & 4 & 0 & 1 & 0 \\ 0 & 4 & 3 & 0 & 0 & 1 \end{array}\right]$$

Enter the matrix in your calculator and find reduced row echelon form.





To convert to fractions push **MATH** **Frac**



The inverse matrix is:

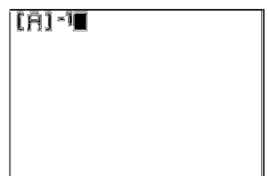
$$A^{-1} = \begin{bmatrix} \frac{7}{4} & \frac{3}{4} & -1 \\ -\frac{3}{4} & -\frac{3}{4} & 1 \\ 1 & 1 & -1 \end{bmatrix}$$

Method 2:

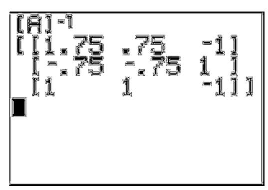
Enter the matrix in your calculator as matrix *A*.

From the home screen push **2nd** x^{-1} **to select your matrix.**

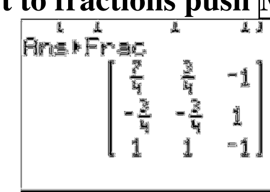
Then push x^{-1}



Then push **ENTER**



To convert to fractions push **MATH** **Frac**



The inverse matrix is:

$$A^{-1} = \begin{bmatrix} \frac{7}{4} & \frac{3}{4} & -1 \\ -\frac{3}{4} & -\frac{3}{4} & 1 \\ 1 & 1 & -1 \end{bmatrix}$$

Practice Exercises E
Find the inverse of the following matrices.

1. $\begin{bmatrix} -1 & -1 & -1 \\ 4 & 5 & 0 \\ 0 & 1 & -3 \end{bmatrix}$

2. $\begin{bmatrix} 1 & -1 & 1 \\ 0 & -2 & 1 \\ -2 & -3 & 0 \end{bmatrix}$

3. $\begin{bmatrix} -2 & 1 & 3 \\ 1 & 2 & -2 \\ 0 & 1 & -1 \end{bmatrix}$

4. $\begin{bmatrix} 1 & 2 & -1 \\ 2 & -1 & 3 \\ -1 & 0 & 1 \end{bmatrix}$

5. $\begin{bmatrix} 2 & 3 & -1 \\ -1 & 0 & 4 \\ 0 & 1 & 1 \end{bmatrix}$

6. $\begin{bmatrix} 5 & 0 & 2 \\ 2 & 2 & 1 \\ -3 & 1 & -1 \end{bmatrix}$

Using the Inverse to Solve a System of Linear Equations

If $AX = B$ has a unique solution, then $X = A^{-1}B$. Where A is the coefficient matrix, X is the column variable matrix, and B is the column solution matrix.

$$\text{Given: } \begin{aligned} ax+by+cz&=d \\ ex+fy+gz&=h \\ kx+my+nz&=p \end{aligned} \quad \text{then, } A=\begin{bmatrix} a & b & c \\ e & f & g \\ k & m & n \end{bmatrix},\ X=\begin{bmatrix} x \\ y \\ z \end{bmatrix},\ \text{and } B=\begin{bmatrix} d \\ h \\ p \end{bmatrix}$$

Example:

$x+y=3$ $-x+3y+4z=-3$ $4y+3z=2$	
$A=\begin{bmatrix} 1 & 1 & 0 \\ -1 & 3 & 4 \\ 0 & 4 & 3 \end{bmatrix},\ X=\begin{bmatrix} x \\ y \\ z \end{bmatrix},\ B=\begin{bmatrix} 3 \\ -3 \\ 2 \end{bmatrix}$	Identify the *A, X,* and *B* matrices
$A^{-1}=\begin{bmatrix} \frac{7}{4} & \frac{3}{4} & -1 \\ -\frac{3}{4} & -\frac{3}{4} & 1 \\ 1 & 1 & -1 \end{bmatrix}$	Find the inverse of *A*
$X=\begin{bmatrix} \frac{7}{4} & \frac{3}{4} & -1 \\ -\frac{3}{4} & -\frac{3}{4} & 1 \\ 1 & 1 & -1 \end{bmatrix}\begin{bmatrix} 3 \\ -3 \\ 2 \end{bmatrix}$	Use $X = A^{-1}B$ to find the solution.
The solution is: $\begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}$ Written as: (1, 2, -2)	$x=1$ $y=2$ $z=-2$

Practice Exercises F
Solve using the inverse matrix.

1.
$$\begin{aligned} 2x+6y+6z&=8 \\ 2x+7y+6z&=10 \\ 2x+7y+7z&=9 \end{aligned}$$

2.
$$\begin{aligned} x+2y+5z&=2 \\ 2x+3y+8z&=3 \\ -x+y+2z&=3 \end{aligned}$$

3.
$$\begin{aligned} x-y+z&=8 \\ 2y-z&=-7 \\ 2x+3y&=1 \end{aligned}$$

4.
$$\begin{aligned} x-6y+3z&=11 \\ 2x-7y+3z&=14 \\ 4x-12y+5z&=25 \end{aligned}$$

5.
$$\begin{aligned} x-y+z&=-6 \\ 4x+2y+z&=9 \\ 4x-2y+z&=-3 \end{aligned}$$

6.
$$\begin{aligned} y+2z&=0 \\ -x+y&=1 \\ 2x-y+z&=-1 \end{aligned}$$

7.
$$\begin{aligned} 3x-2y+z&=-2 \\ 4x-5y+3z&=-9 \\ 2x-y+5z&=-5 \end{aligned}$$

8.
$$\begin{aligned} x-y&=1 \\ 6x+y+20z&=14 \\ y+3z&=1 \end{aligned}$$

9.
$$\begin{aligned} x+3y+4z&=-3 \\ x+2y+3z&=-2 \\ x+4y+3z&=-6 \end{aligned}$$

Unit 2 Cluster 2b (F.IF.8b), Unit 3 Cluster 1b (A.SSE.1b), Unit 3 Cluster 2c (A.SSE.3c)
Forms and Uses of Exponential Functions

Cluster 2: Analyzing functions using different representations

2.2.2b Use properties of exponents to interpret expressions for exponential functions

Cluster 1: Interpreting the structure of expressions

3.1.1b Interpret complicated expressions by looking at one or more of their parts separately (focus on exponential functions with rational exponents using mainly square roots and cube roots)

Cluster 2: Writing expressions in equivalent forms to solve problems

3.2.1c Use properties of exponents to rewrite exponential functions

VOCABULARY

An **exponential function** is a function of the form $f(x) = ab^x$ where a and b are constants and $a \neq 0$, $b > 0$, and $b \neq 1$.

Exponential functions can also be of the form $A = P(1+r)^t$. This is the **simplified interest formula**. Each part of the formula has a specific meaning. The principal, *P*, is the original amount of money that is deposited. The interest rate, *r*, is expressed as a decimal and represents the growth rate of the investment. Time, *t*, is the number of years that the money remains in the account. The amount, *A*, after *t* years can be calculated by using the formula.

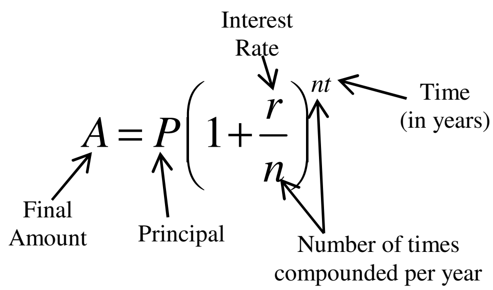
Example:

Austin deposits \$450 into a savings account with a 2.5% interest rate. How much money will be in the account after 5 years?

$A = P(1+r)^t$	$P = \$450$ $r = 2.5\% = 0.025$ $t = 5$ years
$A = 450(1+0.025)^5$	Substitute the known values into the equation
$A = 450(1.025)^5$ $A = 509.1336958$	Evaluate.
Austin will have \$509.13 in his account after 5 years.	

VOCABULARY

Interest is commonly assessed multiple times throughout the year. This is referred to as **compound interest** because the interest is compounded or applied more than once during the year. The formula is $A = P\left(1+\frac{r}{n}\right)^{nt}$ where n is the number of times that the interest is compounded during the year.



Example:

Cyndi and Derek just got married. They received a lot of money at their wedding and decided to invest \$1000 in a Dream CD. A CD is a certified account that pays a fixed interest rate for a specified length of time. Cyndi and Derek chose to do a 3 year CD with a 0.896% interest rate compounded monthly. How much money will they have in 3 years?

$A = P\left(1+\frac{r}{n}\right)^{nt}$	$P = \$1000$ $r = 0.896\% = 0.00896$ $n = 12$ times $t = 3$ years
$A = 1000\left(1+\frac{0.00896}{12}\right)^{12\cdot 3}$	Substitute the known values into the equation
$A = 1000\left(1+\frac{0.00896}{12}\right)^{36}$ $A = 1027.234223$	Simplify the exponent and evaluate.
Cyndi and Derek will have \$1027.23 after 3 years.	

VOCABULARY

There is another interest formula where the interest is assessed continuously. It is called **continuous interest**. The formula is $A = Pe^{rt}$. It uses the same *P*, *r*, and *t* from the other interest formulas but it also utilizes the Euler constant *e*. Similar to pi, *e* is an irrational number. It is defined to be $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e \approx 2.718281828$. It is also called the natural base.

Example:

Eva invested \$750 in a savings account with an interest rate of 1.2% that is compounded continuously. How much money will be in the account after 7 years?

$A = Pe^{rt}$	$P = \$750$ $r = 1.2\% = 0.012$ $t = 7$ years
$A = 750e^{0.012\cdot 7}$	Substitute the known values into the equation
$A = 750e^{0.084}$ $A = 815.7216704$	Simplify the exponent and evaluate.
Eva will have \$815.72 after 7 years.	

VOCABULARY

There are times when the value of an item decreases by a fixed percent each year. This can be modeled by the formula $A = P(1-r)^t$ where *P* is the initial value of the item, *r* is the rate at which its value decreases, and *A* is the value of the item after *t* years.

Example:

Jeff bought a new car for \$27,500. The car's value decreases by 8% each year. How much will the car be worth in 15 years?

$A = P(1-r)^t$	$P = \$27,500$ $r = 8\% = 0.08$ $t = 15$ years
$A = 27,500(1-0.08)^{15}$	Substitute the known values into the equation
$A = 27,500(0.92)^{15}$ $A = 7873.178612$	Evaluate.
Jeff should sell the car for at least \$7873.18.	

Practice Exercises A

Use the compound and continuous interest formulas to solve the following. Round to the nearest cent.

1. Bobbi and Gregg are investing \$10,000 in a money market account that pays 5.5% interest quarterly. How much money will they have after 5 years?

2. Joshua put \$5,000 in a special savings account for 10 years. The account had an interest rate of 6.5% compounded continuously. How much money does he have?

3. Analeigh is given the option of investing \$12,000 for 3 years at 7% compounded monthly or at 6.85% compounded continuously. Which option should she choose and why?

4. Mallory purchased a new Road Glide Ultra motorcycle for \$22,879. Its value depreciates 15% each year. How much could she sell it for 8 years later?

Example:

Emily invested \$1250 after 2 years she had \$1281.45. What was the interest rate, if the interested was assessed once a year?

$A=P(1+r)^t$	The interest is assessed only once a year; use the simplified interest formula.
$A=P(1+r)^t$	$A=\$1281.45$ $P=\$1250$ $t=2$ years
$1281.45=1250(1+r)^2$	Substitute the known values into the equation
$\frac{1281.45}{1250}=(1+r)^2$	Isolate the squared term.
$\sqrt{\frac{1281.45}{1250}}=\sqrt{(1+r)^2}$	Find the square root of each side.
$\pm\sqrt{\frac{1281.45}{1250}}=1+r$ $-1\pm\sqrt{\frac{1281.45}{1250}}=r$ $-1+1.0125=r$ or $-1-1.0125=r$ $0.0125=r$ $\quad$ $-2.0125=r$	Solve for *r*.
The interest rate is positive; therefore it is 0.0125 or 1.25%.	

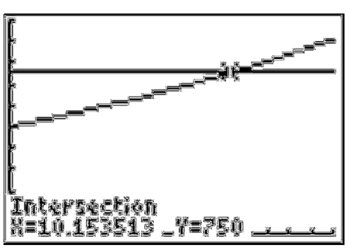
Example:

Sam invested some money in a CD with an interest rate of 1.15% that was compounded quarterly. How much money did Sam invest if he had \$1500 after 10 years?

$A = P\left(1+\frac{r}{n}\right)^{nt}$	The interest is compounded quarterly; use the compound interest formula.
$A = P\left(1+\frac{r}{n}\right)^{nt}$	$A = \$1500$ $r = 1.15\% = 0.015$ $n = 4$ times $t = 10$ years
$1500 = P\left(1+\frac{0.015}{4}\right)^{4\cdot 10}$	Substitute the known values into the equation
$1500 = P(1+0.00375)^{40}$ $1500 = P(1.00375)^{40}$ $\frac{1500}{(1.00375)^{40}} = P$	Isolate *P*.
$1291.424221 = P$	Evaluate.
Ten years ago Sam invested \$1291.42.	

Example:

Suzie received \$500 for her birthday. She put the money in a savings account with 4% interest compounded monthly. When will she have \$750?

$A = P\left(1+\frac{r}{n}\right)^{nt}$	$P = \$500$ $r = 4\% = 0.04$ $n = 12$ times
$A = 500\left(1+\frac{0.04}{12}\right)^{12t}$	Substitute the known values into the equation
	$A = \$750$ Put the interest formula in Y1 and \$750 in Y2. Graph the two equations and use 2nd, Trace, intersect to find their intersection.
Suzie will have \$750 in 10.154 years.	

Practice Exercises B

Solve.

1. Jace invested \$12,000 in a 3-year Dream CD with interest compounded annually. At the end of the 3 years, his CD is worth \$12,450. What was the interest rate for the CD?

2. Jaron has a savings account containing \$5,000 with interest compounded annually. Two years ago, it held \$4,500. What was the interest rate?

3. Lindsey needs to have \$10,000 for the first semester of college. How much does she have to invest in an account that carries an 8.5% interest rate compounded monthly in order to reach her goal in 4 years?

4. If Nick has \$20,000 now, how long will it take him to save \$50,000 in an account that carries an interest of 5.83% compounded continuously?

VOCABULARY

When interest is assessed more than once in the year, the effective interest rate is actually higher than the interest rate. The **effective interest rate** is equivalent to the annual simple rate of interest that would yield the same amount as compounding after 1 year.

	Annual Rate	**Effective Rate**
Annual compounding	10%	10%
Semiannual compounding	10%	10.25%
Quarterly compounding	10%	10.381%
Monthly compounding	10%	10.471%
Daily compounding	10%	10.516%
Continuous compounding	10%	10.517%

Example:

\$1000 is put into a savings account with 5% interest compounded quarterly. What is the effective interest rate?

$A = P\left(1+\frac{r}{n}\right)^{nt}$	$P = \$1000$ $r = 5\% = 0.05$ $n = 4$ times
$A = 1000\left(1+\frac{0.05}{4}\right)^{4t}$	Substitute the known values into the equation

$A=1000(1+0.0125)^{4t}$ $A=1000(1.0125)^{4t}$	Simplify what is inside the parentheses.
$A=1000(1.0125^4)^t$ $A=1000(1.050945337)^t$	Use properties of exponents to rewrite the function so that is to the power of *t*.
$A=1000(1+0.050945337)^t$	Rewrite it so that it is 1 plus the interest rate. This is the simplified interest formula.
In the new simplified interest formula $r=0.0509=5.09\%$	

Example:

If $1000 were put into a savings account that paid 5% interest compounded continuously, what would the monthly interest rate be?

$A=Pe^{rt}$	$P=\$1000$ $r=5\%=0.05$
$A=1000e^{0.05t}$	Substitute the known values into the equation.
$A=1000(e^{0.05})^t$ $A=1000(1.051271096)^t$	Use properties of exponents to rewrite the function so that is to the power of *t*.
$A=1000(1.051271096^{1/12})^{12t}$ $A=1000(1.004175359)^{12t}$ $A=1000(1+0.004175359)^{12t}$	Remember that *t* is the number of **years** the money is being invested. It is necessary to multiply *t* by 12 to convert it to months. Using algebra rules, we must also divide by 12 so that the equation does not change since $\frac{12}{12}=1$. Perform the division inside the parenthesis so the interest rate is affected. Rewrite the information in the parenthesis so that it is 1 plus the interest rate. This is the simplified interest formula.
In the monthly interest rate is $r=0.00418=0.418\%$	

Practice Exercises C

Find the monthly rate or effective interest rate.

1. If \$2,500 is invested in an account with an interest rate of 7.23% compounded semi-annually, what is the effective rate?

2. If \$7,700 is invested in an account with an interest rate of 9% compounded quarterly, what is the monthly interest rate?

3. If \$235,000 is invested in an account with an interest rate of 22.351% compounded monthly, what is the effective rate?

4. If \$550 is invested in an account with an interest rate of 45.9% compounded annually, what is the monthly interest rate?

Exponential Growth and Decay

VOCABULARY

Money is not the only real world phenomenon that can be modeled with an exponential function. Other phenomena such as populations, bacteria, radioactive substances, electricity, and temperatures can be modeled by exponential functions.

A quantity that grows by a fixed percent at regular intervals is said to have **exponential growth**. The formula for **uninhibited growth** is $A(t) = A_0 e^{kt}$, $k > 0$, where A_0 is the original amount, t is the time and k is the growth constant. This formula is similar to the continuous interest formula $A = Pe^{rt}$. Both formulas are continuously growing or growing without any constraints.

$$A(t) = A_0 e^{kt}$$

Time

Growth Constant

Final amount

Initial Amount

Example:

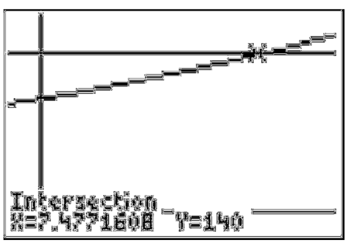
A colony of bacteria that grows according to the law of uninhibited growth is modeled by the function $A(t) = 100e^{0.045t}$, where *A* is measured in grams and *t* is measured in days.

(a) Determine the initial amount of bacteria.

(b) What is the growth constant of the bacteria?

(c) What is the population after 7 days?

(d) How long will it take for the population to reach 140 grams?

a. $A(t) = 100e^{0.045t}$	A_0 is the initial amount and in the equation $A_0 = 100$, therefore the initial amount is 100 grams.
b. $A(t) = 100e^{0.045t}$	k is the growth constant and in the equation $k = 0.045$, therefore the growth rate is 0.045/day.
c. $A(t) = 100e^{0.045t}$ $A(7) = 100e^{0.045(7)}$ $A(7) = 137.026$	Substitute 7 in for time, *t,* then evaluate. After 7 days, there will be 137.026 grams of bacteria.
d. 	$A(t) = 140$ Put the exponential growth equation in Y1 and 140 in Y2. Graph the two equations and use 2nd, Trace, intersect to find their intersection.
It will take 7.477 days for the bacteria to grow to 140 grams.	

VOCABULARY

A quantity that decreases by a fixed percent at regular intervals is said to have **exponential decay**. The formula for **uninhibited decay** is $A(t) = A_0e^{kt}$, $k < 0$ where A_0 is the original amount, *t* is the time and *k* is the constant rate of decay.

The decay formula is the same as the growth formula. The only difference is that when $k > 0$ the amount increases over time making the function have exponential growth and when $k < 0$ the amount decreases over time making the function have exponential decay.

Example:

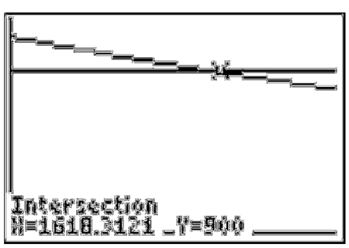
A dinosaur skeleton was found in Vernal, Utah. Scientists can use the equation $A(t) = 1100e^{-0.000124t}$, where A is measured in kilograms and t is measured in years, to determine the amount of carbon remaining in the dinosaur. This in turn helps to determine the age of the dinosaur bones.

(a) Determine the initial amount of carbon in the dinosaur bones.

(b) What is the growth constant of the carbon?

(c) How much carbon is left after 5,600 years?

(d) How long will it take for the carbon to reach 900 kilograms?

a. $A(t) = 1100e^{-0.000124t}$	A_0 is the initial amount and in the equation $A_0 = 1{,}100$, therefore the initial amount is 1,100 kilograms.
b. $A(t) = 1100e^{-0.000124t}$	k is the growth constant and in the equation $k = -0.000124$. Since k is negative, therefore the carbon is decaying at a rate 0.000124/year.
c. $A(t) = 1100e^{-0.000124t}$ $A(5600) = 1100e^{-0.000124(5600)}$ $A(5600) = 549.311$	Substitute 5600 in for time, *t,* then evaluate. After 5600 years, there will be 549.311 kilograms of carbon remaining.
d. 	$A(t) = 900$ Put the exponential growth equation in Y1 and 900 in Y2. Graph the two equations and use 2^{nd}, Trace, intersect to find their intersection.
It will take 1,618.312 years for the carbon to decrease to 900 kilograms.	

Practice Problems D

Solve.

1. India is one of the fastest growing countries in the world. $A(t) = 574e^{0.026t}$ describes the population of India in millions t years after 1974.

   a. What was the population in 1974?
   b. Find the growth constant.
   c. What will the population be in 2030?
   d. When will India's population be 1,624 million?

2. The amount of carbon-14 in an artifact can be modeled by $A(t) = 16e^{-0.000121t}$, where A is measured in grams and t is measured in years.

   a. How many grams of carbon-14 were present initially?
   b. Find the growth constant.
   c. How many grams of carbon-14 will be present after 5,715 years?
   d. When will there be 4 grams of carbon-14 remaining?

3. Medical research indicates that the risk of having a car accident increases exponentially as the concentration of alcohol in the blood increases. The risk is modeled by $R(x) = 6e^{12.77x}$ where R is the percent of risk of having a car accident and x is the blood-alcohol concentration.

   a. What is the risk of having an accident with blood-alcohol concentration of zero?
   b. What is the risk of having an accident with blood-alcohol concentration 0.08?
   c. What is would the blood-alcohol concentration have to be to have a risk factor of 45%?

VOCABULARY

An **exponential function** is a function of the form $A(t) = A_0 b^t = A_0(1+r)^t$ where A_0 is the initial amount, $b = 1 + r$ is the growth factor, and $A_0 \neq 0$, $b > 0$, and $b \neq 1$.

Growth factor = 1 + r

If r < 0, is exponential decay　　　If r > 0, is exponential growth

Example:

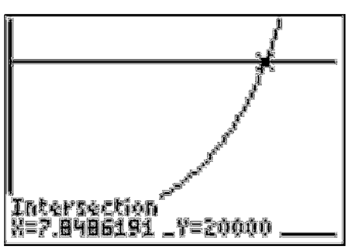
A culture of bacteria obeys the law of uninhibited growth. Initially there were 500 bacteria present. After 1 hour there are 800 bacteria.

a. Identify the growth rate.

b. Write an equation to model the growth of the bacteria

c. How many bacteria will be present after 5 hours?

d. How long is it until there are 20,000 bacteria?

a. $\frac{800}{500} = 1.6$ $b = 1.6 = 1 + 0.6$ $r = 0.6$	Find the common ratio. $\frac{a_{n+1}}{a_n}$ Rewrite b in the form of $1 + r$. Identify the growth rate. $r = 0.6 = 60\%$ The bacterial is increasing at a rate of 60% each hour.
b. $A(t) = A_0 b^t$ $A(t) = 500(1.6)^t$	$A_0 = 500$ $b = 1.6$ Substitute known values into the equation.
c. $A(t) = 500(1.6)^t$ $A(5) = 500(1.6)^5$ $A(5) = 5242.88$	Substitute $t = 5$ into the equation and evaluate. Round down to the nearest whole number. There are 5,242 bacteria present after 5 hours.
d. 	$A(t) = 20,000$ Graph this equation and your equation in Y1 and Y2. Use 2^{nd}, Trace, intersect to find the intersection. There will be 20,000 bacteria after 7.849 hours.

Example:

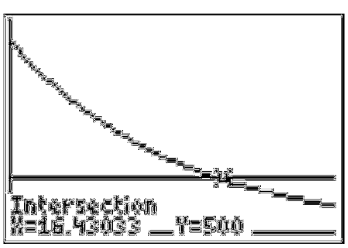
Michael bought a new laptop for $1,800 last year. A month after he purchased it, the price dropped to $1,665.

a. Identify the growth rate.

b. Write an equation to model the value of the computer.

c. What will the value of the computer be after 9 months?

d. When will the value of the computer be $500?

a. $\frac{1800}{1665} = 0.925$ $b = 0.925 = 1 - 0.075$ $r = -0.075$	Find the common ratio. $\frac{a_{n+1}}{a_n}$ Rewrite b in the form of $1 + r$. Identify the growth rate. $r = -0.075 = -7.5\%$ The value of the computer is decreasing at a rate of 7.5% each month.
b. $A(t) = A_0 b^t$ $A(t) = 1800(0.925)^t$	$A_0 = 1800$ $b = 0.925$ Substitute known values into the equation.
c. $A(t) = 1800(0.925)^t$ $A(9) = 1800(0.925)^9$ $A(9) = 892.38$	Substitute $t = 9$ into the equation and evaluate. Round to the nearest cent. After 9 months, the computer is worth $892.38.
d. 	$A(t) = 500$ Graph this equation and your equation in Y1 and Y2. Use 2nd, Trace, intersect to find the intersection. It will take 16.430 months for the computer's value to decrease to $500.

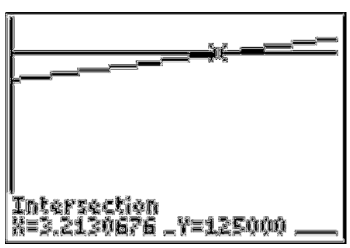
Example:

The population of West Jordan was 106,863 in 2011. The population is growing at a rate of 5% each year.

a. Write an equation to model the population growth.

b. If this trend continues, what will the population be in 2020?

c. How long before the population grows to 125,000 people?

a. $A(t) = A_0(1+r)^t$ $A(t) = 106863(1.05)^t$	$A_0 = 106863$ $r = 5\% = 0.05$ $1 + r = 1 + 0.05 = 1.05$ Substitute known values into the equation.
b. $A(9) = 106863(1.05)^9$ $A(9) = 165779.587$	$t = 2020 - 2011 = 9$ Substitute $t = 9$ into the equation and evaluate. Round down to the nearest whole person. After 9 years, the population of West Jordan is 165,779 people.
c.  	$A(t) = 125000$ Graph this equation and your equation in Y1 and Y2. Use 2^{nd} , Trace, intersect to find the intersection. It will take 16.430 months for the computer's value to decrease to $500.

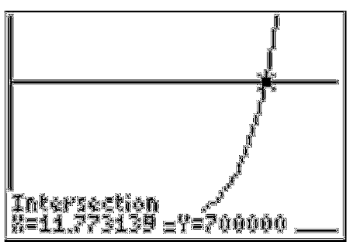
Example:

A culture of 200 bacteria is put in a petri dish and the culture doubles every hour.

a. Write an equation to model the bacteria growth.

b. If this trend continues, how many bacteria will there be in 5 hours?

c. How long before the bacteria population reaches 7000,000?

a. $A(t) = A_0 b^t$ $A(t) = 200(2)^t$	$A_0 = 200$ $r = 100\% = 1.00$ $b = 1 + r = 1 + 1 = 2$ Substitute known values into the equation.
b. $A(t) = 200(2)^t$ $A(5) = 200(2)^5$ $A(5) = 6{,}400$	Substitute $t = 5$ into the equation and evaluate. After 5 hours, there are 6,400 bacteria.
c. 	$A(t) = 700{,}000$ Graph this equation and your equation in Y1 and Y2. Use 2nd , Trace, intersect to find the intersection. It will take 11.773 hours for the bacteria to reach 700,000.

Practice Exercises E

Solve

1. A bird species is in danger of extinction. Last year there were 1,400 birds and today only 1,308 of the birds are alive.

   a. Identify the growth rate.

   b. Write an equation to model the population.

   c. If this trend continues, what will the population be in 10 years?

   d. If the population drops below 100 then the situation will be irreversible. When will this happen?

2. There is a fruit fly in your house. Fruit fly populations triple every day until the food source runs out.

   a. Write an equation to model the fruit fly growth.

   b. If this trend continues, how many fruit flies will there be at the end of 1 week?

   c. How long before the fruit fly population reaches 50,000?

3. In 2003 the population of Nigeria was 124,009,000. It has a growth rate of 3.1%.

   a. Write an equation to model the population growth since 2003.

   b. If this trend continues, what will the population be in 2050?

   c. How long before the population grows to 200,000,000 people?

YOU DECIDE

Utah's population was 2,763,885 in 2010 and 2,817,222 in 2011, find the growth rate. Can the population in Utah continue to grow at this rate indefinitely? Why or why not? Justify your answer.